

Automorphisms of $\mathrm{SU}_n(R)$ and $\mathrm{PGU}_n(R)$ over local rings

Deep H. Makadiya

A lecture based on joint work with Elena I. Bunina

arXiv:2607.22284 (*Automorphisms of Twisted Chevalley Groups of type ${}^2A_\ell$ over Local Rings*)

Aim. Describe all automorphisms of the four groups $\mathrm{EU}_n(R)$, $\mathrm{PEU}_n(R)$, $\mathrm{SU}_n(R)$ and $\mathrm{PGU}_n(R)$, and explain the main idea of the proof.

Time	Topic
0–12 min	Definitions, unitary matrix models, elementary root elements
12–20 min	Standard automorphisms and four classification statements
20–29 min	Reduce an automorphism modulo the maximal ideal
29–48 min	Matrix normalization; one worked calculation; unitary correction
48–57 min	Recover the ring automorphism; pass to the other three groups
57–60 min	Main idea and questions

1 Preliminaries

Let R be a *commutative local ring* with, $1/2 \in R^\times$.

As usual, define

$$\begin{aligned}\mathrm{GL}_n(R) &= \{A \in M_n(R) : \det A \in R^\times\}, \\ \mathrm{SL}_n(R) &= \{A \in \mathrm{GL}_n(R) : \det A = 1\}, \\ \mathrm{PGL}_n(R) &= \mathrm{GL}_n(R)/\{\lambda I_n : \lambda \in R^\times\}.\end{aligned}$$

The last matrix-quotient description uses the local-ring assumption. Denote the projective class of A by $[A]$.

Let $\theta : R \rightarrow R$ be an involution (i.e., $\theta^2 = \mathrm{id}_R$). We use exactly the bar notation from the opening notes:

$$\bar{t} = \theta(t) \quad \text{and} \quad \bar{A} = (\bar{a}_{ij}) \text{ for } A = (a_{ij}) \in M_n(R).$$

Also,

$$R_\theta = \{t \in R : \bar{t} = t\} \quad \text{and} \quad R_\theta^- = \{t \in R : \bar{t} \neq t\}.$$

Then

$$R = R_\theta \oplus R_\theta^-.$$

Fix the *alternating antidiagonal sign matrix*

$$Q = \begin{pmatrix} 0 & \cdots & 0 & 1 \\ 0 & \cdots & -1 & 0 \\ \vdots & \ddots & 0 & 0 \\ (-1)^{n-1} & \cdots & 0 & 0 \end{pmatrix}.$$

Note that $Q^{-1} = Q^t = (-1)^{n-1}Q$.

Define

$$\sigma : \mathrm{SL}_n(R) \rightarrow \mathrm{SL}_n(R), \quad \sigma(A) = Q\bar{A}^{-t}Q^{-1}, \quad A^{-t} = (A^{-1})^t.$$

The same rule descends to $\mathrm{PGL}_n(R)$. Its fixed-point groups are

$$\begin{aligned}\mathrm{SU}_n(R) &= \{A \in \mathrm{SL}_n(R) : \sigma(A) = A\} = \{A \in \mathrm{SL}_n(R) : AQ\bar{A}^t = Q\}, \\ \mathrm{PGU}_n(R) &= \{[A] \in \mathrm{PGL}_n(R) : \sigma([A]) = [A]\} \\ &= \{[A] \in \mathrm{PGL}_n(R) : AQ\bar{A}^t = \lambda Q \text{ for some } \lambda \in R^\times\}.\end{aligned}$$

Remark. The groups $\mathrm{SL}_n(R)$ and $\mathrm{PGL}_n(R)$ are the simply connected and adjoint Chevalley groups of type A_{n-1} over R , respectively. Similarly, $\mathrm{SU}_n(R)$ and $\mathrm{PGU}_n(R)$ are the simply connected and adjoint twisted Chevalley groups of type ${}^2A_{n-1}$ over R , respectively.

2 Elementary subgroups and generators

2.1 Untwisted elementary groups $E_n(R)$ and $\mathrm{PE}_n(R)$

Write $X_\alpha = E_{ij}$ for the matrix unit and put $x_\alpha(t) = I_n + tE_{ij}$ for $i \neq j$, $t \in R$. Define the elementary linear group and its projective image by

$$E_n(R) = \langle e_{ij}(t) \mid i \neq j, t \in R \rangle \leq \mathrm{SL}_n(R), \quad \mathrm{PE}_n(R) = \{[A] \in \mathrm{PGL}_n(R) : A \in E_n(R)\}.$$

As R is local, Gaussian elimination gives $E_n(R) = \mathrm{SL}_n(R)$. Hence $\mathrm{PE}_n(R) \cong E_n(R)/Z(E_n(R))$; this projective elementary group should be distinguished from the whole group $\mathrm{PGL}_n(R)$.

2.2 The A_{n-1} diagram and its two foldings

Let $\Phi = A_{n-1}$ be the root system with simple roots

$$\alpha_i = \varepsilon_i - \varepsilon_{i+1}, \quad 1 \leq i \leq n-1.$$

Its Dynkin diagram is

$$\begin{array}{ccccccc} \circ & \text{---} & \circ & \text{---} & \cdots & \text{---} & \circ & \text{---} & \circ \\ \alpha_1 & & \alpha_2 & & & & \alpha_{n-2} & & \alpha_{n-1} \end{array}$$

Let ρ denote the nontrivial diagram automorphism of Φ , defined on the simple roots by

$$\rho(\alpha_i) = \alpha_{n-i}, \quad 1 \leq i \leq n-1.$$

For each $\alpha \in \Phi$, write $\bar{\alpha} = \rho(\alpha)$ and denote its corresponding twisted root class by $[\alpha]$.

The twisted root classes are of types A_1 , A_1^2 , or A_2 , depending on the action of ρ . The resulting root systems depend on the parity of n , as summarized below:

Case	Folded root system	Types of twisted root classes
$n = 2m$	C_m	A_1 and A_1^2
$n = 2m + 1$	B_m (or BC_m in the nonreduced setting)	A_1^2 and A_2

2.3 Twisted elementary groups $\mathrm{EU}_n(R)$ and $\mathrm{PEU}_n(R)$

Using the Steinberg root elements $x_\alpha(t)$, define

$$\begin{aligned} x_{[\alpha]}(t) &= x_\alpha(t), & [\alpha] &\sim A_1, \quad t \in R_\theta, \\ x_{[\alpha]}(t) &= x_\alpha(t)x_{\bar{\alpha}}(\bar{t}), & [\alpha] &\sim A_1^2, \quad t \in R. \end{aligned}$$

For the A_2 classes (only needed when n is odd), set

$$\mathcal{A}(R) = \{(t_1, t_2) \in R^2 : t_1\bar{t}_1 = t_2 + \bar{t}_2\},$$

and, for a suitable Chevalley-basis sign $N_{\bar{\alpha}, \alpha} = \pm 1$,

$$x_{[\alpha]}((t_1, t_2)) = x_\alpha(t_1)x_{\bar{\alpha}}(\bar{t}_1)x_{\alpha+\bar{\alpha}}(N_{\bar{\alpha}, \alpha}t_2), \quad (t_1, t_2) \in \mathcal{A}(R).$$

For these classes, the root-group parameter law is

$$(t_1, t_2) \oplus (u_1, u_2) = (t_1 + u_1, t_2 + u_2 + \bar{t}_1 u_1).$$

Let $\mathrm{EU}_n(R)$ be subgroup of $\mathrm{SU}_n(R)$ generated by all the above $x_{[\alpha]}(t)$, for all twisted roots and admissible parameters. Its projective elementary counterpart is the image under $\mathrm{SL}_n(R) \rightarrow \mathrm{PGL}_n(R)$:

$$\mathrm{PEU}_n(R) := \{[A] : A \in \mathrm{EU}_n(R)\} \cong \mathrm{EU}_n(R) / \mathrm{Z}(\mathrm{EU}_n(R)).$$

For classes of type A_1 or A_1^2 and a unit parameter, we also have

$$w_{[\alpha]}(t) = x_{[\alpha]}(t) x_{-[\alpha]}(-t^{-1}) x_{[\alpha]}(t), \quad h_{[\alpha]}(t) = w_{[\alpha]}(t) w_{[\alpha]}(1)^{-1}.$$

There are analogous w -elements for type A_2 (see §2.4 of the paper).

Facts. 1. The groups $\mathrm{EU}_n(R)$ and $\mathrm{PEU}_n(R)$ are characteristic subgroups of $\mathrm{SU}_n(R)$ and $\mathrm{PGU}_n(R)$, respectively. In particular,

$$\mathrm{EU}_n(R) \trianglelefteq \mathrm{SU}_n(R) \quad \text{and} \quad \mathrm{PEU}_n(R) \trianglelefteq \mathrm{PGU}_n(R).$$

2. Both $\mathrm{EU}_n(R)$ and $\mathrm{PEU}_n(R)$ are perfect groups.

3. Since R is a local ring, we have

$$\mathrm{SU}_n(R) = \mathrm{TU}_n(R) \mathrm{EU}_n(R) \quad \text{and} \quad \mathrm{PGU}_n(R) = \mathrm{PTU}_n(R) \mathrm{PEU}_n(R).$$

3 Main Theorem

Before stating the main theorem, we introduce the following four types of automorphisms.

1. Strictly inner: $i_g(x) = gxg^{-1}$, with g in the group under consideration.

2. Diagonal: conjugation by a suitable torus element, possibly defined over a ring extension, which preserves the group over R . On root groups it rescales parameters:

$$h(\chi) x_{[\alpha]}(t) h(\chi)^{-1} = x_{[\alpha]}(\chi(\alpha) \cdot t).$$

The symbol $\cdot$ is the appropriate parameter action; for type A_2 , $r \cdot (t_1, t_2) = (rt_1, r\bar{r}t_2)$.

3. Ring: if $\mu \in \mathrm{Aut}(R)$ and $\mu\theta = \theta\mu$, apply μ entrywise; equivalently

$$x_{[\alpha]}(t) \mapsto x_{[\alpha]}(\mu(t)),$$

where $\mu(t_1, t_2) = (\mu(t_1), \mu(t_2))$ for A_2 .

4. Central: $x \mapsto \tau(x)x$, where $\tau : G \rightarrow \mathrm{Z}(G)$ is a homomorphism and the resulting map is an automorphism.

Main theorem (Bunina–Makadiya). Let R be a commutative local ring with $2 \in R^\times$, equipped with the involution θ , and let $n \geq 6$. Then every automorphism is a composition of the following kinds:

Group G	Required types of automorphisms
$G = \mathrm{PEU}_n(R)$	Strictly inner and ring
$G = \mathrm{EU}_n(R)$	Strictly inner, diagonal and ring
$G = \mathrm{PGU}_n(R)$	Strictly inner and ring
$G = \mathrm{SU}_n(R)$	Strictly inner, diagonal, ring and central

The classification for $\mathrm{PEU}_n(R)$ is established first. The remaining results are then obtained using the structural relationships between the elementary and full twisted Chevalley groups.

Remark. Although the proofs for odd and even n involve different computations, their overall strategies are essentially the same. We therefore restrict our discussion to the comparatively simpler case when n is even.

4 Proof idea for $\text{PEU}_n(R)$

Fix

$$G := \text{PEU}_n(R) = E'_{\text{ad},\sigma}(A_{n-1}, R), \quad \varphi \in \text{Aut}(G).$$

We aim to prove that $\varphi = i_c \circ \mu$ with $c \in \text{PGU}_n(R)$ and $\mu \in \text{Aut}(R)$ commuting with θ .

4.1 Step 1: Reduce to the residue field

For each θ -invariant ideal $I \triangleleft R$ define the *principal congruence subgroup inside G*

$$N_I := G \cap \ker(\text{PGU}_n(R) \longrightarrow \text{PGU}_n(R/I)).$$

A normal-subgroup theorem for elementary twisted Chevalley groups gives, for every $H \triangleleft G$, a θ -invariant ideal I such that

$$E'_{\text{ad},\sigma}(A_{n-1}, R, I) \leq H \leq N_I. \quad (\star)$$

Since R is local, any proper ideal I lies in J . In fact, the paper proves:

Key lemma. N_J is the *greatest proper normal subgroup* of G , and

$$G/N_J \cong E'_{\text{ad},\sigma}(A_{n-1}, k) = \text{PEU}_n(k).$$

Why it matters: N_J is characteristic, so φ descends to

$$\bar{\varphi} \in \text{Aut}(\text{PEU}_n(k)).$$

By Steinberg's field theorem, $\bar{\varphi} = i_{\bar{g}} \circ f$, where $\bar{g} \in \text{PGU}_n(k)$ and $f \in \text{Aut}(k)$ commutes with the induced involution. The reduction map $\text{PGU}_n(R) \rightarrow \text{PGU}_n(k)$ is onto, so choose $g_1 \in \text{PGU}_n(R)$ lifting $\bar{g}$. Set

$$\varphi_1 := i_{g_1}^{-1} \circ \varphi.$$

Now $\bar{\varphi}_1 = f$ is *purely a field automorphism* modulo J .

Crucial point: We have *not* assumed that f lifts to an automorphism of R . The ring automorphism will be reconstructed at the end.

4.2 Step 2: Normalize special matrices

We successively change the basis by $g_2, g_3, g_4 \in \text{GL}_n(R)$ with $g_i \equiv I_n \pmod{J}$. In the case $n = 2m$ ($m \geq 3$), the goal is to obtain an isomorphism

$$\varphi_4 = i_{[g_4]}^{-1} \circ i_{[g_3]}^{-1} \circ i_{[g_2]}^{-1} \circ \varphi_1$$

that fixes the three kinds of elements:

$$\boxed{h_{[\alpha]}(-1) \longrightarrow w_{[\alpha]}(1) \longrightarrow x_{[\alpha]}(1)}$$

for every twisted root class $[\alpha]$. The order is intentional:

- First fix the diagonal involutions $h_{[\alpha]}(-1)$ (Section 5 of the paper).
- Next fix the monomial/Weyl matrices $w_{[\alpha]}(1)$ (Section 6).
- Finally fix the unipotent matrices $x_{[\alpha]}(1)$ (Section 7).

The commutation and conjugation relations among these matrices rigidly constrain their images.

One calculation worth showing on the board. After choosing representatives $h_1, \dots, h_m$ of the images of the diagonal involutions, the lifting lemma permits us to arrange

$$h_i^2 = I_n, \quad h_i h_j = h_j h_i, \quad h_i \equiv h_{[\alpha_i]}(-1) \pmod{J}.$$

Write $h_{[\alpha_i]}(-1)e_j = \varepsilon_{ij}e_j$ with $\varepsilon_{ij} \in \{\pm 1\}$. Inductively define

$$e_j^{(r)} = \frac{1}{2}(e_j^{(r-1)} + \varepsilon_{rj}h_re_j^{(r-1)}), \quad e_j^{(0)} = e_j.$$

Then $h_re_j^{(r)} = \varepsilon_{rj}e_j^{(r)}$. Since the h_i commute, earlier eigenvalue conditions remain valid. Moreover $e_j^{(r)} \equiv e_j \pmod{J}$, so these vectors still form an R -basis. The corresponding change-of-basis matrix lies in $\mathrm{GL}_n(R, J)$, where

$$\mathrm{GL}_n(R, J) = \{A \in \mathrm{GL}_n(R) : A \equiv I_n \pmod{J}\}.$$

This is one place where $2 \in R^\times$ is essential.

Change for $n = 2m + 1$ (type ${}^2A_{2m}$). The final simple class is of type A_2 , so $h_{[\alpha]}(-1)$ is normalized only for the A_1^2 classes. The Weyl and root normalizations additionally use

$$w_{[\alpha]} \left(\left(1, \frac{1}{2}\right) \right), \quad x_{[\alpha]} \left(\left(1, \frac{1}{2}\right) \right) \quad ([\alpha] \sim A_2).$$

In this case one also encounters the involution $v_{[\alpha]}(a, b)$ on a three-dimensional block; this is why the odd-dimensional matrix calculation is longer. For a one-hour seminar it is enough to state these modifications, not prove both cases separately.

4.3 Step 3: Why the basis changes are actually unitary

There is a subtlety: the basis changes g_2, g_3, g_4 are initially only in $\mathrm{GL}_n(R, J)$. Their projective classes are *not yet known* to lie in $\mathrm{PGU}_n(R)$. Set $C_1 = g_2g_3g_4$.

Let $u_{[\alpha]} = x_{[\alpha]}(1)$ for classes not of type A_2 , and $u_{[\alpha]} = x_{[\alpha]}(1, \frac{1}{2})$ for type A_2 . The normalized automorphism fixes every $u_{[\alpha]}$. Since conjugation by $[C_1]$ takes each $u_{[\alpha]}$ to an element fixed by σ , a matrix comparison yields

$$Du_{[\alpha]} = \lambda_{[\alpha]}u_{[\alpha]}D, \quad D := Q^{-1}\overline{C_1}^tQC_1, \quad \lambda_{[\alpha]} \equiv 1 \pmod{J}. \quad (\dagger)$$

The scalars define a character on the subgroup generated by these special unipotents. Rank-two commutator relations show $\lambda_{[\alpha]} = 1$ or $\lambda_{[\alpha]}^2 = 1$. In the second case put $\lambda_{[\alpha]} = 1 + a$ with $a \in J$. Then

$$0 = (1 + a)^2 - 1 = a(2 + a).$$

As $2 + a \in R^\times$, we get $a = 0$. Hence every $\lambda_{[\alpha]} = 1$, so D commutes with all $u_{[\alpha]}$.

Matrix-generation lemma (paper, Lemma 3.1): the $u_{[\alpha]}$ generate $M_n(R)$ as a unital R -algebra. Thus their centralizer in $M_n(R)$ is precisely RI_n . It follows that

$$D = \lambda I_n \implies [C_1] \in \mathrm{PGU}_n(R).$$

We have therefore converted the successive *formal matrix normalizations* into a legitimate strictly inner automorphism of G .

4.4 Step 4: Recover the ring automorphism

After normalization, relations with the fixed h -, w - and $x(1)$ -elements force, for every class of type A_1^2 ,

$$\varphi_4(x_{[\alpha]}(t)) = x_{[\alpha]}(\mu(t)) \quad (t \in R)$$

for a *single* map $\mu : R \rightarrow R$, independent of the chosen root class.

Now give two short calculations, using the root-element relations:

(a) Additivity. Since $x_{[\alpha]}(t)x_{[\alpha]}(u) = x_{[\alpha]}(t + u)$,

$$\mu(t + u) = \mu(t) + \mu(u).$$

(b) Multiplicativity. For the first two simple classes (of type A_1^2), the Chevalley commutator relation is

$$[x_{[\alpha_1]}(t), x_{[\alpha_2]}(u)] = x_{[\alpha_1] + [\alpha_2]}(tu).$$

Apply φ_4 and compare parameters to obtain

$$\mu(tu) = \mu(t)\mu(u).$$

Since φ_4 fixes $x_{[\alpha_1]}(1)$, we have $\mu(1) = 1$. Applying the same construction to φ_4^{-1} gives the inverse ring map: $\mu \in \text{Aut}(R)$. A normalized Weyl conjugation sends a parameter t to $-\bar{t}$, so comparing its image under φ_4 gives

$$\mu(\bar{t}) = \overline{\mu(t)} \iff \mu \circ \theta = \theta \circ \mu.$$

Finally, the fixed Weyl matrices propagate the formula to all A_1 classes. In odd dimension, the A_2 parameter law propagates it to $\mathcal{A}(R)$. Thus $\varphi_4 = \mu$ on *all* elementary generators.

Combining all steps,

$$\boxed{\varphi = i_{g_1[C_1]} \circ \mu, \quad g_1[C_1] \in \text{PGU}_n(R), \quad \mu \in \text{Aut}(R), \quad \mu\theta = \theta\mu.}$$

This is the main classification for $\text{PEU}_n(R)$ (Theorem 3.2).

5 Passing from $\text{PEU}_n(R)$ to the other groups

5.1 $\text{EU}_n(R)$: lifting through the center

The natural map

$$\text{EU}_n(R) \longrightarrow \text{PEU}_n(R)$$

is a central quotient. Therefore an automorphism of $\text{EU}_n(R)$ descends to an automorphism of $\text{PEU}_n(R)$, which is already known to be strictly inner times ring. The ring factor lifts entrywise. The inner factor lifts by extending the relevant character of the root lattice to the weight lattice, possibly over a ring extension.

Consequently, the lifted factor is a composition of *strictly inner* and *diagonal* automorphisms. There is no extra central factor on $\text{EU}_n(R)$: this elementary group is perfect, so any homomorphism $\text{EU}_n(R) \rightarrow \text{Z}(\text{EU}_n(R))$ is trivial. This gives the second row of the theorem.

5.2 $\text{PGU}_n(R)$ and $\text{SU}_n(R)$: characteristic elementary subgroup

The elementary subgroup is *characteristic* in the corresponding full twisted group. Restrict $\psi \in \text{Aut}(\text{SU}_n(R))$ to $\text{EU}_n(R)$. Using the elementary result, compose ψ with inverse strictly inner, diagonal and ring automorphisms to arrange that

$$\psi|_{\text{EU}_n(R)} = \text{id}.$$

For $a \in \text{SU}_n(R)$ and $x \in \text{EU}_n(R)$, the element axa^{-1} again belongs to $\text{EU}_n(R)$. Hence

$$\psi(a)x\psi(a)^{-1} = axa^{-1}, \quad a^{-1}\psi(a) \in C_{\text{SU}_n(R)}(\text{EU}_n(R)).$$

The centralizer of the elementary subgroup in the full unitary group equals the center, so

$$\tau(a) := a^{-1}\psi(a) \in \text{Z}(\text{SU}_n(R)).$$

It is readily checked that τ is a homomorphism and $\psi(a) = \tau(a)a$: the remaining automorphism is *central*. This proves the last row.

For the adjoint full group $\text{PGU}_n(R)$ the center is trivial, so the same argument gives no central factor. Furthermore its diagonal automorphisms are strictly inner. This gives the third row.

6 Conclusion and two takeaways

The automorphisms contain no unexpected kind of symmetry. They arise from conjugation, allowed changes of the coefficient ring, diagonal rescaling and (only for the full non-adjoint statement) central twists.

1. **From ring to field and back.** The characteristic congruence kernel modulo J permits a passage to Steinberg’s field result; the parameter-dependent root elements recover a *genuine* automorphism of R without presupposing a lift of the residue-field automorphism.
2. **Rigidity from small matrices.** Commuting involutions, Weyl matrices and unipotents determine all of $M_n(R)$. The final scalar-commutant argument forces the normalizing basis change to be projectively unitary.

Scope. Here $n \geq 6$ (equivalently $\ell \geq 5$). The lower-rank cases need separate normalization arguments. The 63-page paper supplies all the omitted odd/even-rank matrix calculations.

Reference. Elena I. Bunina and Deep H. Makadiya, *Automorphisms of Twisted Chevalley Groups of type ${}^2A_\ell$ ($\ell \geq 5$) over Local Rings*, arXiv:2607.22284 (2026). <https://arxiv.org/abs/2607.22284>