Automorphisms of Twisted Chevalley Groups

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Outline

Chevalley and Twisted Chevalley Groups

Algebraic Groups

Affine variety: A subset $V \subseteq \mathbb{C}^n$ of common zeros of a collection of polynomials $\{f_1, \ldots, f_k\} \subseteq \mathbb{C}[x_1, \ldots, x_n]$.

Morphism: A map $\varphi : X \to Y$ of affine varieties whose coordinate functions are given by polynomials.

A **linear algebraic group** over $\mathbb{C}$ is a group G that is also an affine variety over $\mathbb{C}$, such that the multiplication map $m : G \times G \to G$, given by $(a, b) \mapsto a \cdot b$, and the inverse map $i : G \to G$, given by $x \mapsto x^{-1}$, are morphisms of varieties.

Examples: $\mathbb{C}$, $\mathbb{C}^\times$, $GL_n(\mathbb{C})$, and $SL_n(\mathbb{C})$.

Semisimple Algebraic Groups

A connected linear algebraic group G is said to be
(a) semisimple if every connected solvable normal subgroup of G is trivial.
(b) simple if it is semisimple & has exactly 2 connected normal subgroups.

Fact

Semisimple algebraic groups over $\mathbb{C}$ are classified by their root systems, which determine their type, together with a sublattice of the (fundamental) weight lattice containing the root lattice, which determine their isogeny class.

Let Φ be a (reduced) root system, and let Λ_{ad} and Λ_{sc} denote the corresponding root lattice and (fundamental) weight lattice, respectively.

Consider a lattice Λ_π such that $\Lambda_{\text{ad}} \subseteq \Lambda_\pi \subseteq \Lambda_{\text{sc}}$. The unique (up to isomorphism) semisimple linear algebraic group over $\mathbb{C}$ associated with the pair (Φ, Λ_π) is denoted by $G_\pi(\Phi, \mathbb{C})$.

Examples

The following table presents examples of simple linear algebraic groups over $\mathbb{C}$:

Type of Φ	$G_{sc}(\Phi, \mathbb{C})$	$G_{\mathrm{ad}}(\Phi, \mathbb{C})$
A_ℓ	$\mathrm{SL}_{\ell+1}(\mathbb{C})$	$\mathrm{PSL}_{\ell+1}(\mathbb{C}) \cong \mathrm{PGL}_{\ell+1}(\mathbb{C})$
B_ℓ	$\mathrm{Spin}_{2\ell+1}(\mathbb{C})$	$\mathrm{SO}_{2\ell+1}(\mathbb{C}) = \mathrm{PSO}_{2\ell+1}(\mathbb{C})$
C_ℓ	$\mathrm{Sp}_{2\ell}(\mathbb{C})$	$\mathrm{PSp}_{2\ell}(\mathbb{C})$
D_ℓ	$\mathrm{Spin}_{2\ell}(\mathbb{C})$	$\mathrm{PSO}_{2\ell}(\mathbb{C})$

Fact: $G_{\mathrm{ad}}(\Phi, \mathbb{C}) \cong G_\pi(\Phi, \mathbb{C})/Z(G_\pi(\Phi, \mathbb{C}))$ & $Z(G_\pi(\Phi, \mathbb{C})) \cong \Lambda_\pi/\Lambda_{\mathrm{ad}}$.

Type of Φ	A_ℓ	B_ℓ	C_ℓ	$D_{2\ell+1}$	$D_{2\ell}$	E_6	E_7	E_8	F_4	G_2
$\Lambda_{sc}/\Lambda_{\mathbf{ad}}$	$\mathbb{Z}_{\ell+1}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_4$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_2$	$\{0\}$	$\{0\}$	$\{0\}$

Hence $Z(G_\pi(\Phi, \mathbb{C}))$ is finite, while $Z(G_{\mathrm{ad}}(\Phi, \mathbb{C}))$ is trivial.

For a fixed root system Φ, the groups $G_\pi(\Phi, \mathbb{C})$, associated with lattices Λ_π (such that $\Lambda_{\mathrm{ad}} \subset \Lambda_\pi \subset \Lambda_{sc}$) are isogenous.

Chevalley Groups

Fact: The group $G_\pi(\Phi, \mathbb{C})$ is defined over $\mathbb{Z}$.

Therefore, we can define the R-group $G_\pi(\Phi, R)$ for any commutative ring R with unity. This group is known as the **Chevalley group** of type Φ over R.

If π is such that $\Lambda_\pi = \Lambda_{\text{ad}}$ (resp., $\Lambda_\pi = \Lambda_{sc}$), then $G_\pi(\Phi, R) = G_{\text{ad}}(\Phi, R)$ (resp., $G_\pi(\Phi, R) = G_{sc}(\Phi, R)$) is called an **adjoint Chevalley group** (resp., **universal** (or **simply connected**) **Chevalley group**).

Type of Φ	$G_{sc}(\Phi, R)$	$G_{\text{ad}}(\Phi, R)$
A_ℓ	$\mathrm{SL}_{\ell+1}(R)$	$\mathrm{PGL}_{\ell+1}(R)$
B_ℓ	$\mathrm{Spin}_{2\ell+1}(R)$	$\mathrm{PSO}_{2\ell+1}(R)$
C_ℓ	$\mathrm{Sp}_{2\ell}(R)$	$\mathrm{PSp}_{2\ell}(R)$
D_ℓ	$\mathrm{Spin}_{2\ell}(R)$	$\mathrm{PSO}_{2\ell}(R)$

Elementary Chevalley Groups

Let R be a commutative ring with unity and let Φ be a root system. For any root $\alpha \in \Phi$ and any element $t \in R$, there is an element $x_\alpha(t) \in G_\pi(\Phi, R)$ called **transvection**.

Roughly speaking, "the element $x_\alpha(t)$ is given by exponential of certain nilpotent elements associated with root α of the corresponding Lie algebra".

A subgroup of $G_\pi(\Phi, R)$ generated by $x_\alpha(t)$ $(\alpha \in \Phi, t \in R)$ is called **elementary Chevalley group** of type Φ over R and is denoted by $E_\pi(\Phi, R)$.

Example

Let $G = G_{\mathrm{sc}}(A_\ell, R) = \mathrm{SL}_{\ell+1}(R)$.

- **Root System:**

$$\Phi = \{\epsilon_i - \epsilon_j \mid 1 \leq i \neq j \leq \ell + 1\}.$$

- **Root Elements:** For $\alpha = \epsilon_i - \epsilon_j$, the root vector is $X_\alpha = E_{i,j}$. The corresponding root element is:

$$x_\alpha(t) = I + tX_\alpha = I + tE_{i,j}.$$

- **Elementary Subgroup:** $E_{\mathrm{sc}}(A_\ell, R)$ is the subgroup generated by

$$\{x_\alpha(t) \mid \alpha \in \Phi,\ t \in R\}.$$

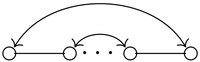
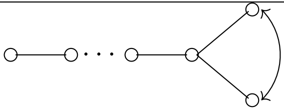
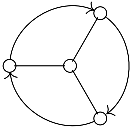
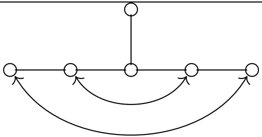
Twisted Root System

Let Φ be a root system. Let Δ and Φ^+ be the simple and positive root systems, respectively. Let W be the corresponding Weyl group.

Assume that Φ is of type A_ℓ $(\ell \geq 1), D_\ell$ $(\ell \geq 4)$ or E_6.

Let ρ be a non-trivial angle preserving permutation of Δ. Then it can be extended to a permutation of Φ.

This ρ induces an automorphism of the corresponding Dynkin diagram.

Root System	Graph Automorphism ρ	Order of ρ
$A_n\,(n \geq 2)$		2
$D_n\,(n \geq 4)$		2
D_4		3
E_6		2

For each $\alpha \in \Phi$, define

$$\hat{\alpha} := \frac{1}{o(\rho)} \sum_{i=1}^{o(\rho)} \rho^i(\alpha),$$

where $o(\rho)$ denotes the order of the automorphism ρ.

Define an equivalence relation "$\equiv$" on Φ by $\alpha \equiv \beta$ iff $\hat{\alpha}$ is a positive multiple of $\hat{\beta}$.

Let Φ_ρ denote the collection of all equivalence classes $[\alpha]$ for all $\alpha \in \Phi$. Then forms a reduced root system.

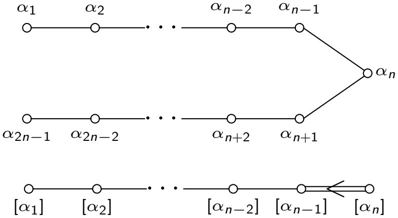
We call Φ_ρ a twisted root system. If type of Φ is X and $o(\rho) = n$ then $\Phi_\rho \sim {}^nX$.

Note that each $[\alpha]$ is a positive root system of type either A_1, A_1^2, A_1^3 or A_2. In this case, we denote $[\alpha] \sim A_1, A_1^2, A_1^3$ or A_2, respectively.

Twisted Root System (continue)

Type of Φ	Φ_ρ	Type of Roots	
		Long	Short
${}^2A_{2n-1}$ $(n \geq 2)$	C_n	A_1	A_1^2
${}^2A_{2n}$ $(n \geq 2)$	B_n	A_1^2	A_2
${}^2D_n$ $(n \geq 4)$	B_{n-1}	A_1	A_1^2
${}^3D_4$	G_2	A_1	A_1^3
${}^2E_6$	F_4	A_1	A_1^2

Example: Φ is of type A_{2n-1} $(n \geq 2)$ and $o(\rho) = 2$. Then Φ_ρ is of type C_n.



Twisted Chevalley Groups

Let $G(R) = G_\pi(\Phi, R)$ be a Chevalley group over a commutative ring R.

Let σ be an automorphism of $G(R)$ of order 2 (or 3) which is the product of a graph automorphism ρ and a ring automorphism θ such that $o(\sigma) = o(\rho) = o(\theta)$.

Define $G_\sigma(R) = \{g \in G(R) \mid \sigma(g) = g\}$. Then clearly $G_\sigma(R)$ is a subgroup of $G(R)$, called a **twisted Chevalley group**.

Write $E'_\sigma(R)$ for the subgroup of $G_\sigma(R)$ generated by $x_{[\alpha]}(t)$ where $[\alpha] \in \Phi_\rho$ and $t \in R_{[\alpha]}$. We call this an **elementary twisted Chevalley group**.

Example

Let $\Phi \sim A_{2m}$.

Simply Connected Case:

- Untwisted group: $G_{\mathrm{sc}}(\Phi, R) = \mathrm{SL}_{2m+1}(R)$
- Involution σ: $A \mapsto Q\bar{A}^{-t}Q^{-1}$, where Q is anti-diagonal with alternating signs $(1, -1, 1, \dots)$.
- Twisted group (Special Unitary):

$$G_{\mathrm{sc},\sigma}(\Phi, R) = \mathrm{SU}_{2m+1}(R) = \{A \in \mathrm{SL}_{2m+1}(R) \mid AQ\bar{A}^t = Q\}.$$

Adjoint Case:

- Untwisted group: $G_{\mathrm{ad}}(\Phi, R) = \mathrm{PGL}_{2m+1}(R)$.
- Twisted group (Projective Unitary):

$$G_{\mathrm{ad},\sigma}(\Phi, R) = \left\{[A] \in \mathrm{PGL}_{2m+1}(R) \;\middle|\; AQ\bar{A}^t = \lambda Q \text{ for some } \lambda \in R^\times\right\}.$$

Identifying Certain Chevalley and Twisted Chevalley Groups

Recall that

$$\sigma = \rho \circ \theta = \theta \circ \rho.$$

We assume throughout that both ρ and θ are non-trivial.

Question: What happens if one of them is trivial?

- **If $\rho = \mathrm{id}$:**

$$G_{\pi,\sigma}(\Phi, R) \cong G_\pi(\Phi, R_\theta).$$

- **If $\theta = \mathrm{id}$:**

$$G_{\pi,\sigma}(\Phi, R) \cong G_\pi(\Phi_\rho, R).$$

- **If $\rho \neq \mathrm{id} \neq \theta$:**
  - If $\Phi_\rho \not\sim {}^3D_4$ and $R = S \times S$, with $\theta(a,b) = (b,a)$, then

$$G_{\pi,\sigma}(\Phi, R) \cong G_\pi(\Phi, S).$$

  - If $\Phi_\rho \sim {}^3D_4$ and $R = S \times S \times S$, with $\theta(a,b,c) = (b,c,a)$, then

$$G_{\pi,\sigma}(\Phi, R) \cong G_\pi(\Phi, S).$$

Automorphisms of Chevalley and Twisted Chevalley Groups

Standard Automorphisms of Untwisted Groups

Set $G = G(R) := G_\pi(\Phi, R)$ or $E(R) := E_\pi(\Phi, R)$.

Inner: Let $S \supseteq R$. If $g \in N_{G(S)}(G)$, then the conjugation map i_g of G is called *inner automorphism*. It is called *strictly inner* if $g \in G$.

Graph: An automorphism of the form $e_1\rho_1 + \cdots + e_k\rho_k$ is called *graph automorphism*, where e_i are mutually orthogonal idempotents and ρ_i are automorphism induced by symmetries of the Dynkin diagram of Φ.

Ring: Any $\mu \in \mathrm{Aut}(R)$ induces a *ring automorphism* of $G(R)$, also denoted by μ.

Central: A homomorphism $\tau : G(R) \to Z(G(R))$ induces an endomorphism $x \to \tau(x)x$ of $G(R)$, also denoted by τ. If τ is an automorphism, it is called a *central automorphism*. For adjoint and elementary groups, these are trivial.

An automorphism of G is **standard** if it is a composition of inner, graph, ring, and central automorphisms.

Classifications of Automorphisms of Untwisted Groups

- **Classical groups over fields:** Initiated by Schreier and van der Waerden (1928) for PSL_n over fields, and further developed by Dieudonné, Hua, O'Meara, and many others.
- **Chevalley groups over fields:** Steinberg (1960) provided the classification over finite fields, later extended to arbitrary fields by Humphreys (1969).
- **Over various classes of commutative rings:** E. Abe, A. Klyachko, V. Petechuk and others extended these classifications to various classes of rings. Finally, E. I. Bunina and co-authors finalized the generalization to arbitrary commutative rings under mild conditions.

Theorem (E. I. Bunina)

Let R be a commutative ring and let Φ be an irreducible root system of rank ≥ 2. Assume that $1/2 \in R$ for types A_2, B_ℓ $(\ell \geq 2)$, C_ℓ $(\ell \geq 3)$, and F_4, and that $1/6 \in R$ for type G_2. Then every automorphism of $G_\pi(\Phi, R)$ and of $E_\pi(\Phi, R)$ is standard.

Automorphism of Twisted Groups

- Over fields, the automorphisms of twisted Chevalley groups are described by classical results of Steinberg and Humphreys.
- Much less is explore over general commutative rings.
- We have recently started a project to classify automorphisms of twisted Chevalley groups:

[A] Groups of type ${}^2A_\ell$ ($\ell \geq 5$) over local rings $\longrightarrow$ submitted

[B] Groups of type ${}^2A_\ell$ ($\ell = 3, 4$) over local rings $\longrightarrow$ almost submitted

[C] Groups of type ${}^2D_\ell$ ($\ell \geq 4$) over local rings $\longrightarrow$ submitted

[D] Groups of type ${}^2E_6$ over local rings $\longrightarrow$ under final edits

[E] Groups of type ${}^3D_4$ over local rings $\longrightarrow$ under final edits

[F] Groups of all types over commutative rings $\longrightarrow$ under final edits

Main Theorems

Type of Groups	**Condition on R**	$\mathrm{Aut}(\mathbf{E})$ & $\mathrm{Aut}(\mathbf{G})$	**Appear in**
$^2A_\ell$ ($\ell \geq 5$)	Local with $1/2$	Standard	[A]
	CR with $1/2$		[F]
$^2A_\ell$ ($\ell = 3, 4$)	Local with $1/2$	Non-standard	[B]
	CR with $1/2$		[F]
$^2D_\ell$ ($\ell \geq 5$)	Local with $1/2$	Standard	[C]
	CR with $1/2$		[F]
$^2D_4$	Local with $1/2$	Standard	[C]
	CR with $1/2$	Standard with graph	[F]
$^3D_4$	Local with $1/6$	Special Standard	[E]
	CR with $1/6$		[F]
$^2E_6$	Local with $1/2$	Standard	[D]
	CR with $1/2$		[F]

Standard: inner, ring and central.

Standard with graph: inner, ring, central and graph.

Outline of the Proof for ${}^2A_{2m-1}$ over Local Rings

Statements

Theorem

Let R be a local ring with $1/2 \in R$ and $G = E'_{\mathrm{ad},\sigma}(A_{2\ell}, R)$. Then every automorphism of G is a composition of a strictly inner automorphism and a ring automorphism.

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Theorem

Let R be a local ring with $1/2 \in R$ and $G = G_{\pi,\sigma}(A_{2\ell}, R)$. Then every automorphism of G is a composition of an inner, a central, and a ring automorphism.

Modulo the Jacobson Radical

For simplicity, write

$$E(R) = E'_{\mathrm{ad},\sigma}(A_{2\ell}, R), \qquad G(R) = G_{\mathrm{ad},\sigma}(A_{2\ell}, R).$$

Let J be the Jacobson radical of R, and let $k := R/J$ be the residue field.

Let $\varphi \in \mathrm{Aut}(E(R))$. Then φ induces an automorphism $\bar{\varphi}$ of $E(k)$ via the following commutative diagram:

$$\begin{tikzcd} E(R) \arrow[r, "\varphi"] \arrow[d, "\lambda_J"'] & E(R) \arrow[d, "\lambda_J"] \\ E(k) \arrow[r, "\bar{\varphi}"'] & E(k) \end{tikzcd}$$

By the classical result of Steinberg,

$$\bar{\varphi} = i_{\bar{g}} \circ f,$$

where $\bar{g} \in G(k)$ and f is a field automorphism of $E(k)$.

Reduction to Key Steps

Since the natural map

$$\lambda_J : G(R) \to G(k)$$

is surjective, we can lift $\bar{g} \in G(k)$ to $g_1 \in G(R)$.

Set $\varphi_1 := i_{g_1}^{-1} \circ \varphi$. Then $\varphi_1 \in \mathrm{Aut}(E(R))$ such that $\bar{\varphi}_1 = f$.

- **Step 1:** There exist an element $g_2 \in \mathrm{GL}_{\ell+1}(R, J)$ such that the map $\varphi_2 := i_{[g_2]^{-1}} \circ \varphi_1$ satisfies

$$\varphi_2 \left(h_{[\alpha]}(-1) \right) = h_{[\alpha]}(-1) \qquad \text{for all } [\alpha] \in \Phi_\rho.$$

- **Step 2:** There exist an element $g_3 \in \mathrm{GL}_{\ell+1}(R, J)$ such that the map $\varphi_3 := i_{[g_3]^{-1}} \circ \varphi_2$ satisfies

$$\varphi_3 \left(w_{[\alpha]}(1) \right) = w_{[\alpha]}(1) \qquad \text{for all } [\alpha] \in \Phi_\rho.$$

- **Step 3:** There exist an element $g_4 \in \mathrm{GL}_{\ell+1}(R, J)$ such that the map $\varphi_4 := i_{[g_4]^{-1}} \circ \varphi_3$ satisfies

$$\varphi_4\left(x_{[\alpha]}(1)\right) = x_{[\alpha]}(1) \qquad \text{for all } [\alpha] \in \Phi_\rho.$$

Set $C_1 := g_2 g_3 g_4 \in \mathrm{GL}_{\ell+1}(R, J)$. Then

$$\varphi_1 = i_{[C_1]} \circ \varphi_4.$$

Lemma

$[C_1] \in G_{\mathrm{ad},\sigma}(\Phi, R)$.

- **Step 4:** There exists an automorphism μ of the ring R such that $\mu \circ \theta = \theta \circ \mu$ and

$$\varphi_4(x_{[\alpha]}(t)) = x_{[\alpha]}(\mu(t)) \qquad \text{for all } [\alpha] \in \Phi_\rho \text{ and } t \in R_{[\alpha]}.$$

Assuming that Steps 1–4 have been established, the map φ_4 is induced by a ring automorphism $\mu \in \mathrm{Aut}(R)$. Following the standard notation, we also denote φ_4 by μ.

Therefore,

$$\varphi = i_{g_1} \circ \varphi_1 = i_{g_1} \circ i_{[C_1]} \circ \mu.$$

Setting

$$g = g_1[C_1] \in G_{\mathrm{ad},\sigma}(\Phi, R),$$

we obtain

$$\varphi = i_g \circ \mu,$$

where i_g is a strictly inner automorphism and μ is a ring automorphism.

This completes the proof of the theorem.

Thank You!