Triangular and Unitriangular Factorization of $SL(n, R)$ and $SU(n, R)$

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Basic Notations

Let R be a commutative ring with unity.

Let $G = SL(n, R) = \{A \in M(n, R) \mid \det(A) = 1\}$.

Let $\{\epsilon_i \mid i = 1, \dots, n\}$ denote the standard basis of $\mathbb{R}^n$.

Define:

$$
\begin{aligned}
\Phi &= \{\epsilon_i - \epsilon_j \mid 1 \le i \ne j \le n\} &\longrightarrow&\quad \text{Root system of } SL(n, R) \\
\Phi^+ &= \{\epsilon_i - \epsilon_j \mid 1 \le i < j \le n\} &\longrightarrow&\quad \text{Set of positive roots} \\
\Phi^- &= \{\epsilon_i - \epsilon_j \mid 1 \le j < i \le n\} &\longrightarrow&\quad \text{Set of negative roots}
\end{aligned}
$$

Clearly,

$$\Phi = \Phi^+ \cup \Phi^-, \quad \text{with} \quad \Phi^- = -\Phi^+.$$

The elements $x_\alpha(t)$

For $\alpha = \epsilon_i - \epsilon_j \in \Phi$ and $t \in R$, define

$$x_\alpha(t) = I_n + tE_{i,j} = \begin{pmatrix} 1 & 0 & \dots & \dots & 0 & 0 \\ 0 & 1 & \dots & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & t & \vdots & \vdots \\ \vdots & \vdots & & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \dots & 1 & 0 \\ 0 & 0 & \dots & \dots & 0 & 1 \end{pmatrix}.$$

Clearly, $x_\alpha(t) \in SL(n, R)$.

Observations:

1. If $\alpha \in \Phi^+$, then $x_\alpha(t)$ is upper triangular.
2. If $\alpha \in \Phi^-$, then $x_\alpha(t)$ is lower triangular.

The Elementary Subgroup

Define

$$\begin{aligned}
U^{+}(n,R) &= \langle x_\alpha(t) \mid \alpha \in \Phi^{+},\ t \in R\rangle \\
&= \text{the set of all upper triangular matrices in } SL(n,R) \\
&\quad\ \text{with 1's on the diagonal,} \\
U^{-}(n,R) &= \langle x_\alpha(t) \mid \alpha \in \Phi^{-},\ t \in R\rangle \\
&= \text{the set of all lower triangular matrices in } SL(n,R) \\
&\quad\ \text{with 1's on the diagonal,} \\
T(n,R) &= \text{the set of all diagonal matrices in } SL(n,R).
\end{aligned}$$

Finally, define

$$E(n,R) = \langle U^{+}(n,R), U^{-}(n,R)\rangle = \langle x_\alpha(t) \mid \alpha \in \Phi,\ t \in R\rangle,$$

which is called the **elementary subgroup** of $SL(n,R)$.

Relation between $SL(n, R)$ and $E(n, R)$

Fact

If $R = k$ is a field, then $E(n, k) = SL(n, k)$.

Questions:

1. Does the equality $E(n, R) = SL(n, R)$ hold for a general ring R?
   Answer: No. (For example, take $R = \mathbb{Z}[\sqrt{-5}]$.)
2. Find a class of rings for which this equality holds.
   Answer: Semi-local rings, Euclidean rings, etc.

For $n \geq 3$, the subgroup $E(n, R)$ is normal in $SL(n, R)$, and the quotient

$$K_1(n, R) := SL(n, R)/E(n, R)$$

is called the K_1-*functor* of $SL(n, R)$.

Bruhat Decomposition

Fact

If $R = k$ is a field, then the *Bruhat decomposition* holds for $G = SL(n, R)$, namely:

$$G = \bigcup_{\overline{w} \in W} BwB,$$

where:

- $W = N(T(n, R))/T(n, R)$ is the Weyl group,
- $B = B(n, R) = T(n, R)\, U^{+}(n, R)$ is a Borel subgroup.

Question: Does the Bruhat decomposition hold for general rings?

Answer: It does not hold in general (not even for a local ring).

Gauss Decomposition

One can consider a modified version of the Bruhat decomposition as follows:

$$G = \bigcup_{\overline{w} \in W} BwB = \bigcup_{\overline{w} \in W} (TU^+)\, w\, (TU^+)$$
$$\subset (TU^+)(U^+U^-U^+)(TU^+) = TU^+U^-U^+.$$

On the other hand, it is clear that

$$TU^+U^-U^+ \subset G.$$

Therefore,

$$G = TU^+U^-U^+.$$

Such a decomposition is called the *Gauss decomposition* of G.

Conclusion: Bruhat decomposition $\implies$ Gauss decomposition.

Question: For which class of rings does the Gauss decomposition hold for the group G?

Triangular and Unitriangular Factorizations

In general, one may ask: for which classes of rings does the following decomposition hold?

1. **Triangular factorization:**

$$G = T\, U^+ U^- U^+ U^- \dots U^{\pm}.$$

Remark

It is well known that $T = T(n, R) \subset (U^+ U^- U^+ U^- U^+)^{n-1}$.

2. **Unitriangular factorization:**

$$G = U^+ U^- U^+ U^- \dots U^{\pm}.$$

Remark

It is more natural to split this problem into the following two questions:

1. *When does* $G = E$ *hold?*
2. *When does* E *admit such a decomposition?*

Triangular and Unitriangular Factorization for $SL(n, R)$

Definition

A ring R is said to satisfy the *stable range one* condition (SR_1) if, for any $a, b \in R$ with $aR + bR = R$, there exists $z \in R$ such that $a + bz \in R^\times$.

Examples: Semi-local rings (e.g., fields, local rings).

Theorem (N. A. Vavilov, B. Sury, and A. V. Smolensky, 2011–12)

Let R be a commutative ring with unity. If R satisfies the (SR_1) condition, then:

1. $E(n, R) = T(n, R)\, U^+(n, R)\, U^-(n, R)\, U^+(n, R), \quad n \geq 2;$
2. $E(n, R) = U^+(n, R)\, U^-(n, R)\, U^+(n, R)\, U^-(n, R), \quad n \geq 2.$

Conversely, if (1) *holds for some $n \geq 2$, then R satisfies the* (SR_1) *condition.*

Remarks on Known Results

This type of result also holds for classical groups such as

$$\mathrm{Spin}(2n+1,R), \quad \mathrm{Sp}(2n,R), \quad \mathrm{Spin}(2n,R), \quad \mathrm{SU}(2n,R).$$

For the case of $\mathrm{SU}(2n+1,R)$, the result has not been proven in full generality.

In 2013, A. Smolensky explored unitriangular decompositions of $\mathrm{SU}(2n+1,R)$ in the case where $R=\mathbb{C}$ as well as when R is a finite field.

The group $SU(n, R)$

Let $\theta : R \to R$ be a ring automorphism of order 2.

For $r \in R$, write $\bar{r} = \theta(r)$, and for a matrix $A = (a_{ij}) \in SL(n, R)$, set $\bar{A} = (\bar{a}_{ij})$.

Let $G = SL(n, R)$ and define

$$\sigma : G \longrightarrow G, \qquad A \longmapsto J\,(\bar{A}^t)^{-1}\,J^{-1},$$

where

$$J = \begin{pmatrix} 0 & 0 & \dots & 0 & 0 & -1 \\ 0 & 0 & \dots & 0 & 1 & 0 \\ 0 & 0 & \dots & -1 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \pm 1 & 0 & \dots & 0 & 0 & 0 \end{pmatrix}.$$

The group $SU(n, R)$ (continue)

Define

$$SU(n,R) := G_\sigma = \{A \in G \mid \sigma(A) = A\} = \{A \in SL(n,R) \mid A^t J \bar{A} = J\}.$$

Also, set

$$\begin{aligned}
U_\sigma^+(n,R) &= U^+(n,R) \cap SU(n,R),\\
U_\sigma^-(n,R) &= U^-(n,R) \cap SU(n,R),\\
E_\sigma'(n,R) &= \langle U_\sigma^+(n,R),\ U_\sigma^-(n,R)\rangle,\\
T_\sigma'(n,R) &= \text{the set of all diagonal matrices in } E_\sigma'(n,R),\\
E_\sigma(n,R) &= E(n,R) \cap SU(n,R),\\
T_\sigma(n,R) &= T(n,R) \cap SU(n,R).
\end{aligned}$$

Fact: If n is even, then

$$T_\sigma(n,R) = T_\sigma'(n,R).$$

For odd n, this need not hold in general.

The group $SU(3, R)$

By definition,

$$SU(3,R) = \left\{ A \in SL(3,R) \mid A^t J \bar{A} = J \right\},$$

where

$$J = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}.$$

Note that, the matrix

$$\begin{pmatrix} 1 & t & u \\ 0 & 1 & s \\ 0 & 0 & 1 \end{pmatrix} \in SU(3,R) \iff s = \bar{t} \quad \text{and} \quad t\bar{t} = u + \bar{u}.$$

Define

$$\mathcal{A}(R) = \{(t,u) \in R^2 \mid t\bar{t} = u + \bar{u}\}.$$

Moreover, set

$$\mathcal{A}(R)^\times = \{(t,u) \in \mathcal{A}(R) \mid u \in R^\times\}.$$

For $(t, u) \in \mathcal{A}(R)$, define

$$x_+(t,u) = \begin{pmatrix} 1 & t & u \\ 0 & 1 & \overline{t} \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad x_-(t,u) = \begin{pmatrix} 1 & 0 & 0 \\ \overline{t} & 1 & 0 \\ u & t & 1 \end{pmatrix}.$$

Then both $x_+(t,u)$ and $x_-(t,u)$ lie in $SU(3,R)$. Note that

$$U_\sigma^+(3,R) = \{x_+(t,u) \mid (t,u) \in \mathcal{A}(R)\}, \quad \text{and}$$
$$U_\sigma^-(3,R) = \{x_-(t,u) \mid (t,u) \in \mathcal{A}(R)\}.$$

For every $r \in R^*$, define

$$h(r) = \begin{pmatrix} r & 0 & 0 \\ 0 & \overline{r}r^{-1} & 0 \\ 0 & 0 & \overline{r}^{-1} \end{pmatrix}.$$

Then $h(r) \in SU(3,R)$ and

$$T_\sigma(3,R) = \{h(r) \mid r \in R^\times\}.$$

Note that, for $r \in R^\times$ and $(t, u) \in \mathcal{A}(R)$, we have

$$h(r)\, x_+(t,u)\, h(r)^{-1} = x_+\left(r^2\bar{r}^{-1}t,\ r\bar{r}u\right), \tag{H1}$$

$$h(r)\, x_-(t,u)\, h(r)^{-1} = x_-\left(r\bar{r}^{-2}t,\ r^{-1}\bar{r}^{-1}u\right). \tag{H2}$$

Next, for every $r \in R^\times$, define

$$w(r) = \begin{pmatrix} 0 & 0 & r \\ 0 & -r^{-1}\bar{r} & 0 \\ \bar{r}^{-1} & 0 & 0 \end{pmatrix}.$$

Then we have $w(r) \in SU(3, R)$. For $r \in R^\times$ and $(t, u) \in \mathcal{A}(R)$, we have

$$w(r)\, x_+(t,u)\, w(r)^{-1} = x_-\left(-r\bar{r}^{-2}t,\ (r\bar{r})^{-1}u\right), \tag{W1}$$

$$w(r)\, x_-(t,u)\, w(r)^{-1} = x_+\left(-r^2\bar{r}^{-1}t,\ r\bar{r}u\right). \tag{W2}$$

Moreover, for $r, s \in R^\times$ we have

$$h(r) = w(r)w(1), \tag{HW1}$$

$$h(r)\, w(s)\, h(r)^{-1} = w(r\bar{r}s), \tag{HW2}$$

$$w(r)\, h(s)\, w(r)^{-1} = h(\bar{s}^{-1}). \tag{HW3}$$

Special Stable Range One Condition

Recall that a ring R satisfies the (SR_1) *condition* if, for any $a, b \in R$ with $aR + bR = R$, there exists $z \in R$ such that $a + zb \in R^\times$.

The condition $aR + bR = R$ can equivalently be stated as: the vector $(a, b) \in R^2$ can be completed to a matrix in $SL(2, R)$.

Definition

Let R be a commutative ring equipped with an involution θ.

1. A vector $(a, b, c) \in R^3$ is called *special unitary completable* if there exists a matrix $A \in SU(3, R)$ whose first row is (a, b, c).
2. The ring R satisfies the *special stable range one condition* (SSR_1) if, for any special unitary completable vector $(a, b, c) \in R^3$, there exists a pair $(z_1, z_2) \in \mathcal{A}(R)$ such that

$$a + bz_1 + cz_2 \in R^\times,$$

where $\mathcal{A}(R) = \{(t, u) \in R^2 \mid t\bar{t} = u + \bar{u}\}$.

Examples

Examples of rings satisfying the (SR_1) condition:

- Semilocal rings (in particular, local rings and fields).

Examples of rings satisfying the (SSR_1) condition:

- Let R be a semilocal ring with an involution θ. Suppose that $\overline{\mathfrak{m}} = \mathfrak{m}$ for all $\mathfrak{m} \in \mathrm{Max}(R)$ and that $\mathcal{A}(R)^\times \neq \emptyset$. Then R satisfies the (SSR_1) condition.
- Let R be a local ring with an involution θ such that $\mathcal{A}(R)^\times \neq \emptyset$. Then R satisfies the (SSR_1) condition.
- Every field (with an involution) satisfies the (SSR_1) condition.

Triangular Factorization of $SU(3, R)$

Theorem (S. Garge and D. Makadiya)

Let R be a commutative ring with an involution θ. The following conditions are equivalent:

1. *R satisfies (SSR_1) condition,*
2. $SU(3, R) = T_\sigma(3, R)U_\sigma^-(3, R)U_\sigma^+(3, R)U_\sigma^-(3, R)$,
3. $SU(3, R) = T_\sigma(3, R)U_\sigma^+(3, R)U_\sigma^-(3, R)U_\sigma^+(3, R)$.

Outline of the proof.

(1) $\implies$ (2). Let $A = (a_{ij}) \in SU(3, R)$.

Then the vector (a_{11}, a_{12}, a_{13}) is a special unitary completable.

Since R satisfies (SSR_1), there exists $(z_1, z_2) \in \mathcal{A}(R)$ such that

$$a_{11} + a_{12}z_1 + a_{13}z_2 \in R^\times.$$

Define the matrix $B = (b_{ij})$ by

$$B = A\, x_-(\bar{z}_1, z_2) \in SU(3, R).$$

In particular, we have $b_{11} = a_{11} + a_{12}z_1 + a_{13}z_2 \in R^\times$.

Next, define the matrix $C = (c_{ij})$ by

$$C = x_- \left(-\frac{\overline{b_{21}}}{\overline{b_{11}}},\ -\frac{b_{31}}{b_{11}} + \frac{b_{21}\overline{b_{21}}}{b_{11}\overline{b_{11}}} \right) B \in SU(3, R).$$

We observe that $c_{21} = c_{31} = 0$. Since $C \in SU(3, R)$, it follows that $c_{32} = 0$ as well. By the same reason, C must have the form

$$C = \begin{pmatrix} c_{11} & * & * \\ 0 & (c_{11})^{-1}\bar{c}_{11} & * \\ 0 & 0 & (\bar{c}_{11})^{-1} \end{pmatrix} = h(c_{11})\, x_+(*, *),$$

where $x_+(*, *) \in U_\sigma^+(3, R)$.

Therefore, we conclude that

$$x_-\left(-\frac{\overline{b_{21}}}{\overline{b_{11}}},\ -\frac{b_{31}}{b_{11}}+\frac{b_{21}\overline{b_{21}}}{b_{11}\overline{b_{11}}}\right)\cdot A\cdot x_-(\bar{z}_1, z_2)=h(c_{11})\,x_+(*,*).$$

Rewriting A accordingly, we obtain:

$$\begin{aligned}
A &= x_-(*,*)\cdot h(*)\cdot x_+(*,*)\cdot x_-(*,*)\\
&= h(*)\cdot\big(h(*)^{-1}x_-(*,*)h(*)\big)\cdot x_+(*,*)\cdot x_-(*,*)\\
&= h(*)\cdot x_-(*,*)\cdot x_+(*,*)\cdot x_-(*,*)\\
&\in T_\sigma(3,R)\,U_\sigma^-(3,R)\,U_\sigma^+(3,R)\,U_\sigma^-(3,R),
\end{aligned}$$

as required.

(2) $\implies$ (1). Let $(a, b, c) \in R^3$ be a special unitary completable vector. Then, by definition, there exists a matrix $A \in SU(3, R)$ of the form

$$A = \begin{pmatrix} a & b & c \\ * & * & * \\ * & * & * \end{pmatrix}.$$

By assumption, there exists $r \in R^\times$ and $(z_1, z_2) \in \mathcal{A}(R)$ such that

$$A = h(r)\, x_-(*, *)\, x_+(*, *)\, x_-(-\bar{z}_1, \bar{z}_2).$$

Multiplying both sides on the right by $x_-(\bar{z}_1, z_2)$ yields

$$A\, x_-(\bar{z}_1, z_2) = h(r)\, x_-(*, *)\, x_+(*, *).$$

Comparing the $(1, 1)$-entry on both sides, we obtain

$$a + bz_1 + cz_2 = r \in R^\times,$$

as required.

(2) $\iff$ (3) : The equivalence of these statements follows directly from the following identities:

$$
\begin{aligned}
w(1)\, SU(3,R)\, w(1)^{-1} &= SU(3,R) && \text{since } w(1) \in SU(3,R), \\
w(1)\, T_\sigma(3,R)\, w(1)^{-1} &= T_\sigma(3,R) && \text{by equation (HW3)}, \\
w(1)\, U_\sigma^{+}(3,R)\, w(1)^{-1} &= U_\sigma^{-}(3,R) && \text{by equation (W1)}, \\
w(1)\, U_\sigma^{-}(3,R)\, w(1)^{-1} &= U_\sigma^{+}(3,R) && \text{by equation (W2)}.
\end{aligned}
$$

This completes the proof. $\square$

Triangular Factorization of $SU(2n+1, R)$

Theorem (S. Garge and D. Makadiya)

Let R be a commutative ring with an involution θ. Assume that

1. *R satisfies (SSR_1) condition, if $n = 1$,*
2. *R satisfies both the (SR_1) and (SSR_1) condition, if $n \geq 2$.*

Then

$$\begin{aligned} E'_\sigma(2n+1, R) &= T'_\sigma(n, R)U^+_\sigma(n, R)U^-_\sigma(n, R)U^+_\sigma(n, R), \\ &= T'_\sigma(n, R)U^-_\sigma(n, R)U^+_\sigma(n, R)U^-_\sigma(n, R). \end{aligned}$$

θ-complete ring

By definition, we have

$$T'_\sigma(2n+1,R) \subseteq T_\sigma(2n+1,R).$$

Question

Under what conditions does the equality

$$T'_\sigma(2n+1,R) = T_\sigma(2n+1,R)$$

hold?

Define

$$\mathcal{B}_1(R) = \{u \in R \mid \exists t \in R \text{ such that } (t,u) \in \mathcal{A}(R)^\times\},$$
$$\mathcal{B}_k(R) = \{u \in R \mid \exists u_1,\ldots,u_k \in \mathcal{B}_1(R) \text{ such that } u = u_1 \cdots u_k\},$$
$$C_{even}(R) = \bigcup_{\ell \in \mathbb{N}} \mathcal{B}_{2\ell}(R).$$

θ-complete ring (continue)

Definition

Let R be a commutative ring with an involution θ.

1. We say that R is θ-complete if $\mathcal{C}_{even}(R) = R^{\times}$.
2. If $\exists k \in \mathbb{N}$ such that $\mathcal{C}_{even}(R) = \mathcal{B}_{2k}(R)$, then we say that R has finite $\mathcal{C}$-length. The minimal such k for which this equality holds is called the $\mathcal{C}$-length of R.

Example

- Every field is θ-complete with $\mathcal{C}$-length 1.
- Let R be a local ring. Assume that $2 \in R^{\times}$ and $R^{\times} \cap R_{\theta}^{-} \neq \emptyset$. Then R is θ-complete with $\mathcal{C}$-length ≤ 2.

Unitriangular Factorization of $SU(2n+1, R)$

Proposition

Let R be a θ-complete ring with $\mathcal{C}$-length k. Then

$$T_\sigma(2n{+}1, R) = T'_\sigma(2n{+}1, R) \subseteq (U^+_\sigma(2n{+}1, R)U^-_\sigma(2n{+}1, R))^{k+1}, \ (n \geq 1).$$

Theorem

Let R be a commutative ring with an involution θ. Assume that

1. *R is θ-complete with $\mathcal{C}$-length k and satisfies the (SSR_1) condition if $n = 1$,*
2. *R is θ-complete with $\mathcal{C}$-length k and satisfies both the (SR_1) and (SSR_1) condition if $n \geq 2$.*

Then

$$E'_\sigma(2n+1, R) = (U^+_\sigma(2n+1, R)U^-_\sigma(2n+1, R))^{k+1}U^+_\sigma(2n+1, R).$$

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Thank You!