Triangular and Unitriangular Factorization of $SL(n, R)$ and $SU(n, R)$

Deep H. Makadiya

(joint work with Prof. Shripad M. Garge)

Department of Mathematics, IIT Bombay

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Basic Notations

Let R be a commutative ring with unity.

Let $G = SL(n,R) = \{A \in M(n,R) \mid \det(A) = 1\}$.

Let $\{\epsilon_i \mid i = 1, \ldots, n\}$ denote the standard basis of $\mathbb{R}^n$.

Define:

$$\begin{aligned}
\Phi &= \{\epsilon_i - \epsilon_j \mid 1 \leq i \neq j \leq n\} & \longrightarrow & \quad \text{Root system of } SL(n,R) \\
\Phi^+ &= \{\epsilon_i - \epsilon_j \mid 1 \leq i < j \leq n\} & \longrightarrow & \quad \text{Set of positive roots} \\
\Phi^- &= \{\epsilon_i - \epsilon_j \mid 1 \leq j < i \leq n\} & \longrightarrow & \quad \text{Set of negative roots}
\end{aligned}$$

Clearly,

$$\Phi = \Phi^+ \cup \Phi^-, \quad \text{with} \quad \Phi^- = -\Phi^+.$$

The elements $x_\alpha(t)$

For $\alpha = \epsilon_i - \epsilon_j \in \Phi$ and $t \in R$, define

$$x_\alpha(t) = I_n + tE_{i,j} = \begin{pmatrix} 1 & 0 & \dots & \dots & 0 & 0 \\ 0 & 1 & \dots & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & t & \vdots & \vdots \\ \vdots & \vdots & & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \dots & 1 & 0 \\ 0 & 0 & \dots & \dots & 0 & 1 \end{pmatrix}.$$

Clearly, $x_\alpha(t) \in SL(n, R)$.

Observations:

1. If $\alpha \in \Phi^+$, then $x_\alpha(t)$ is upper triangular.
2. If $\alpha \in \Phi^-$, then $x_\alpha(t)$ is lower triangular.

The Elementary Subgroup

Define

$$\begin{aligned} U^+(n,R) &= \langle x_\alpha(t) \mid \alpha \in \Phi^+,\ t \in R\rangle \\ &= \text{the set of all upper triangular matrices in } SL(n,R) \\ &\quad\ \text{with 1's on the diagonal}, \\ U^-(n,R) &= \langle x_\alpha(t) \mid \alpha \in \Phi^-,\ t \in R\rangle \\ &= \text{the set of all lower triangular matrices in } SL(n,R) \\ &\quad\ \text{with 1's on the diagonal}, \\ T(n,R) &= \text{the set of all diagonal matrices in } SL(n,R). \end{aligned}$$

Finally, define

$$E(n,R) = \langle U^+(n,R), U^-(n,R)\rangle = \langle x_\alpha(t) \mid \alpha \in \Phi,\ t \in R\rangle,$$

which is called the **elementary subgroup** of $SL(n,R)$.

Relation between $SL(n, R)$ and $E(n, R)$

Fact

If $R = k$ is a field, then $E(n, k) = SL(n, k)$.

Questions:

1. Does the equality $E(n, R) = SL(n, R)$ hold for a general ring R?
   Answer: No. (For example, take $R = \mathbb{Z}[\sqrt{-5}]$.)
2. Find a class of rings for which this equality holds.
   Answer: Semi-local rings, Euclidean rings, etc.

For $n \geq 3$, the subgroup $E(n, R)$ is normal in $SL(n, R)$, and the quotient

$$K_1(n, R) := SL(n, R)/E(n, R)$$

is called the K_1-*functor* of $SL(n, R)$.

Bruhat Decomposition

Fact

If $R = k$ is a field, then the *Bruhat decomposition* holds for $G = SL(n, R)$, namely:

$$G = \bigcup_{\overline{w} \in W} BwB,$$

where:

- $W = N(T(n, R))/T(n, R)$ is the Weyl group,
- $B = B(n, R) = T(n, R)\, U^{+}(n, R)$ is a Borel subgroup.

Question: Does the Bruhat decomposition hold for general rings?

Answer: It does not hold in general (not even for a local ring).

Gauss Decomposition

One can consider a modified version of the Bruhat decomposition as follows:

$$G = \bigcup_{\overline{w} \in W} BwB = \bigcup_{\overline{w} \in W} (TU^+)\, w\, (TU^+)$$
$$\subset (TU^+)(U^+U^-U^+)(TU^+) = TU^+U^-U^+.$$

On the other hand, it is clear that

$$TU^+U^-U^+ \subset G.$$

Therefore,

$$G = TU^+U^-U^+.$$

Such a decomposition is called the *Gauss decomposition* of G.

Conclusion: Bruhat decomposition $\implies$ Gauss decomposition.

Question: For which class of rings does the Gauss decomposition hold for the group G?

Triangular and Unitriangular Factorizations

In general, one may ask: for which classes of rings does the following decomposition hold?

1. **Triangular factorization:**

$$G = T\, U^+ U^- U^+ U^- \dots U^\pm.$$

Remark

It is well known that $T = T(n, R) \subset (U^+ U^- U^+ U^- U^+)^{n-1}$.

2. **Unitriangular factorization:**

$$G = U^+ U^- U^+ U^- \dots U^\pm.$$

Remark

It is more natural to split this problem into the following two questions:

1. *When does* $G = E$ *hold?*
2. *When does* E *admit such a decomposition?*

Triangular and Unitriangular Factorization for $SL(n, R)$

Definition

A ring R is said to satisfy the *stable range one* condition (SR_1) if, for any $a, b \in R$ with $aR + bR = R$, there exists $z \in R$ such that $a + bz \in R^{\times}$.

Examples: Semi-local rings (e.g., fields, local rings).

Theorem (N. A. Vavilov, B. Sury, and A. V. Smolensky, 2011–12)

Let R be a commutative ring with unity. If R satisfies the (SR_1) condition, then:

1. $E(n, R) = T(n, R)\, U^{+}(n, R)\, U^{-}(n, R)\, U^{+}(n, R), \quad n \geq 2;$
2. $E(n, R) = U^{+}(n, R)\, U^{-}(n, R)\, U^{+}(n, R)\, U^{-}(n, R), \quad n \geq 2.$

Conversely, if (1) holds for some $n \geq 2$, then R satisfies the (SR_1) condition.

Remarks on Known Results

This type of result also holds for classical groups such as

$$\mathrm{Spin}(2n+1, R), \quad \mathrm{Sp}(2n, R), \quad \mathrm{Spin}(2n, R), \quad \mathrm{SU}(2n, R).$$

For the case of $\mathrm{SU}(2n+1, R)$, the result has not been proven in full generality.

In 2013, A. Smolensky explored unitriangular decompositions of $\mathrm{SU}(2n+1, R)$ in the case where $R = \mathbb{C}$ as well as when R is a finite field.

The group $SU(n, R)$

Let $\theta : R \to R$ be a ring automorphism of order 2.

For $r \in R$, write $\bar{r} = \theta(r)$, and for a matrix $A = (a_{ij}) \in SL(n, R)$, set $\bar{A} = (\bar{a}_{ij})$.

Let $G = SL(n, R)$ and define

$$\sigma : G \longrightarrow G, \qquad A \longmapsto J\,(\bar{A}^t)^{-1}\,J^{-1},$$

where

$$J = \begin{pmatrix} 0 & 0 & \dots & 0 & 0 & -1 \\ 0 & 0 & \dots & 0 & 1 & 0 \\ 0 & 0 & \dots & -1 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \pm 1 & 0 & \dots & 0 & 0 & 0 \end{pmatrix}.$$

The group $SU(n, R)$ (continue)

Define

$$SU(n,R) := G_\sigma = \{A \in G \mid \sigma(A) = A\} = \{A \in SL(n,R) \mid A^t J \bar{A} = J\}.$$

Also, set

$$\begin{aligned} U_\sigma^+(n,R) &= U^+(n,R) \cap SU(n,R), \\ U_\sigma^-(n,R) &= U^-(n,R) \cap SU(n,R), \\ E_\sigma'(n,R) &= \langle U_\sigma^+(n,R),\ U_\sigma^-(n,R) \rangle, \\ T_\sigma'(n,R) &= \text{the set of all diagonal matrices in } E_\sigma'(n,R), \\ E_\sigma(n,R) &= E(n,R) \cap SU(n,R), \\ T_\sigma(n,R) &= T(n,R) \cap SU(n,R). \end{aligned}$$

Fact: If n is even, then

$$T_\sigma(n,R) = T_\sigma'(n,R).$$

For odd n, this need not hold in general.

The group $SU(3, R)$

By definition,

$$SU(3,R) = \left\{A \in SL(3,R) \mid A^t J \bar{A} = J\right\},$$

where

$$J = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}.$$

Note that, the matrix

$$\begin{pmatrix} 1 & t & u \\ 0 & 1 & s \\ 0 & 0 & 1 \end{pmatrix} \in SU(3,R) \iff s = \bar{t} \quad \text{and} \quad t\bar{t} = u + \bar{u}.$$

Define

$$\mathcal{A}(R) = \{(t,u) \in R^2 \mid t\bar{t} = u + \bar{u}\}.$$

Moreover, set

$$\mathcal{A}(R)^\times = \{(t,u) \in \mathcal{A}(R) \mid u \in R^\times\}.$$

For $(t, u) \in \mathcal{A}(R)$, define

$$x_+(t,u) = \begin{pmatrix} 1 & t & u \\ 0 & 1 & \overline{t} \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad x_-(t,u) = \begin{pmatrix} 1 & 0 & 0 \\ \overline{t} & 1 & 0 \\ u & t & 1 \end{pmatrix}.$$

Then both $x_+(t,u)$ and $x_-(t,u)$ lie in $SU(3,R)$. Note that

$$U_\sigma^+(3,R) = \{x_+(t,u) \mid (t,u) \in \mathcal{A}(R)\}, \quad \text{and}$$
$$U_\sigma^-(3,R) = \{x_-(t,u) \mid (t,u) \in \mathcal{A}(R)\}.$$

For every $r \in R^*$, define

$$h(r) = \begin{pmatrix} r & 0 & 0 \\ 0 & \overline{r}r^{-1} & 0 \\ 0 & 0 & \overline{r}^{-1} \end{pmatrix}.$$

Then $h(r) \in SU(3,R)$ and

$$T_\sigma(3,R) = \{h(r) \mid r \in R^\times\}.$$

Note that, for $r \in R^\times$ and $(t,u) \in \mathcal{A}(R)$, we have

$$h(r)\, x_+(t,u)\, h(r)^{-1} = x_+\left(r^2\overline{r}^{-1}t,\ r\overline{r}u\right), \tag{H1}$$

$$h(r)\, x_-(t,u)\, h(r)^{-1} = x_-\left(r\overline{r}^{-2}t,\ r^{-1}\overline{r}^{-1}u\right). \tag{H2}$$

Next, for every $r \in R^\times$, define

$$w(r) = \begin{pmatrix} 0 & 0 & r \\ 0 & -r^{-1}\overline{r} & 0 \\ \overline{r}^{-1} & 0 & 0 \end{pmatrix}.$$

Then we have $w(r) \in SU(3,R)$. For $r \in R^\times$ and $(t,u) \in \mathcal{A}(R)$, we have

$$w(r)\, x_+(t,u)\, w(r)^{-1} = x_-\left(-r\overline{r}^{-2}t,\ (r\overline{r})^{-1}u\right), \tag{W1}$$

$$w(r)\, x_-(t,u)\, w(r)^{-1} = x_+\left(-r^2\overline{r}^{-1}t,\ r\overline{r}u\right). \tag{W2}$$

Moreover, for $r, s \in R^\times$ we have

$$h(r) = w(r)w(1), \tag{HW1}$$

$$h(r)\, w(s)\, h(r)^{-1} = w(r\overline{r}s), \tag{HW2}$$

$$w(r)\, h(s)\, w(r)^{-1} = h(\overline{s}^{-1}). \tag{HW3}$$

Special Stable Range One Condition

Recall that a ring R satisfies the (SR_1) *condition* if, for any $a, b \in R$ with $aR + bR = R$, there exists $z \in R$ such that $a + zb \in R^\times$.

The condition $aR + bR = R$ can equivalently be stated as: the vector $(a, b) \in R^2$ can be completed to a matrix in $SL(2, R)$.

Definition

Let R be a commutative ring equipped with an involution θ.

1. A vector $(a, b, c) \in R^3$ is called *special unitary completable* if there exists a matrix $A \in SU(3, R)$ whose first row is (a, b, c).
2. The ring R satisfies the *special stable range one condition* (SSR_1) if, for any special unitary completable vector $(a, b, c) \in R^3$, there exists a pair $(z_1, z_2) \in \mathcal{A}(R)$ such that

$$a + bz_1 + cz_2 \in R^\times,$$

   where $\mathcal{A}(R) = \{(t, u) \in R^2 \mid t\bar{t} = u + \bar{u}\}$.

Examples

Examples of rings satisfying the (SR_1) condition:

- Semilocal rings (in particular, local rings and fields).

Examples of rings satisfying the (SSR_1) condition:

- Let R be a semilocal ring with an involution θ. Suppose that $\overline{\mathfrak{m}} = \mathfrak{m}$ for all $\mathfrak{m} \in \mathrm{Max}(R)$ and that $\mathcal{A}(R)^{\times} \neq \emptyset$. Then R satisfies the (SSR_1) condition.
- Let R be a local ring with an involution θ such that $\mathcal{A}(R)^{\times} \neq \emptyset$. Then R satisfies the (SSR_1) condition.
- Every field (with an involution) satisfies the (SSR_1) condition.

Triangular Factorization of $SU(3, R)$

Theorem (S. Garge and D. Makadiya)

Let R be a commutative ring with an involution θ. The following conditions are equivalent:

1. *R satisfies (SSR_1) condition,*
2. $SU(3, R) = T_\sigma(3, R)U_\sigma^-(3, R)U_\sigma^+(3, R)U_\sigma^-(3, R)$,
3. $SU(3, R) = T_\sigma(3, R)U_\sigma^+(3, R)U_\sigma^-(3, R)U_\sigma^+(3, R)$.

Outline of the proof.

(1) $\implies$ (2). Let $A = (a_{ij}) \in SU(3, R)$.

Then the vector (a_{11}, a_{12}, a_{13}) is a special unitary completable.

Since R satisfies (SSR_1), there exists $(z_1, z_2) \in \mathcal{A}(R)$ such that

$$a_{11} + a_{12}z_1 + a_{13}z_2 \in R^\times.$$

Define the matrix $B = (b_{ij})$ by

$$B = A\, x_-(\bar{z}_1, z_2) \in SU(3, R).$$

In particular, we have $b_{11} = a_{11} + a_{12}z_1 + a_{13}z_2 \in R^\times$.

Next, define the matrix $C = (c_{ij})$ by

$$C = x_- \left(-\frac{\overline{b_{21}}}{\overline{b_{11}}},\ -\frac{b_{31}}{b_{11}} + \frac{b_{21}\overline{b_{21}}}{b_{11}\overline{b_{11}}} \right) B \in SU(3, R).$$

We observe that $c_{21} = c_{31} = 0$. Since $C \in SU(3, R)$, it follows that $c_{32} = 0$ as well. By the same reason, C must have the form

$$C = \begin{pmatrix} c_{11} & * & * \\ 0 & (c_{11})^{-1}\bar{c}_{11} & * \\ 0 & 0 & (\bar{c}_{11})^{-1} \end{pmatrix} = h(c_{11})\, x_+(*, *),$$

where $x_+(*, *) \in U_\sigma^+(3, R)$.

Therefore, we conclude that

$$x_-\left(-\frac{\overline{b_{21}}}{\overline{b_{11}}},\ -\frac{b_{31}}{b_{11}}+\frac{b_{21}\overline{b_{21}}}{b_{11}\overline{b_{11}}}\right)\cdot A\cdot x_-(\bar{z}_1,z_2)=h(c_{11})\,x_+(*,*).$$

Rewriting A accordingly, we obtain:

$$\begin{aligned}A &= x_-(*,*)\cdot h(*)\cdot x_+(*,*)\cdot x_-(*,*)\\ &= h(*)\cdot\big(h(*)^{-1}x_-(*,*)h(*)\big)\cdot x_+(*,*)\cdot x_-(*,*)\\ &= h(*)\cdot x_-(*,*)\cdot x_+(*,*)\cdot x_-(*,*)\\ &\in T_\sigma(3,R)\,U_\sigma^-(3,R)\,U_\sigma^+(3,R)\,U_\sigma^-(3,R),\end{aligned}$$

as required.

(2) $\implies$ (1). Let $(a, b, c) \in R^3$ be a special unitary completable vector. Then, by definition, there exists a matrix $A \in SU(3, R)$ of the form

$$A = \begin{pmatrix} a & b & c \\ * & * & * \\ * & * & * \end{pmatrix}.$$

By assumption, there exists $r \in R^\times$ and $(z_1, z_2) \in \mathcal{A}(R)$ such that

$$A = h(r)\, x_-(*, *)\, x_+(*, *)\, x_-(-\bar{z}_1, \bar{z}_2).$$

Multiplying both sides on the right by $x_-(\bar{z}_1, z_2)$ yields

$$A\, x_-(\bar{z}_1, z_2) = h(r)\, x_-(*, *)\, x_+(*, *).$$

Comparing the $(1, 1)$-entry on both sides, we obtain

$$a + bz_1 + cz_2 = r \in R^\times,$$

as required.

(2) $\iff$ (3) : The equivalence of these statements follows directly from the following identities:

$$\begin{aligned}
w(1)\,SU(3,R)\,w(1)^{-1} &= SU(3,R) && \text{since } w(1) \in SU(3,R),\\
w(1)\;T_\sigma(3,R)\,w(1)^{-1} &= T_\sigma(3,R) && \text{by equation (HW3)},\\
w(1)\;U_\sigma^{+}(3,R)\,w(1)^{-1} &= U_\sigma^{-}(3,R) && \text{by equation (W1)},\\
w(1)\;U_\sigma^{-}(3,R)\,w(1)^{-1} &= U_\sigma^{+}(3,R) && \text{by equation (W2)}.
\end{aligned}$$

This completes the proof. $\square$

Theorem (S. Garge and D. Makadiya)

Let R be a commutative ring with an involution θ. Assume that

1. *R satisfies (SSR_1) condition, if $n = 1$,*
2. *R satisfies both the (SR_1) and (SSR_1) condition, if $n \geq 2$.*

Then

$$\begin{aligned} E'_\sigma(2n+1, R) &= T'_\sigma(n, R) U^+_\sigma(n, R) U^-_\sigma(n, R) U^+_\sigma(n, R), \\ &= T'_\sigma(n, R) U^-_\sigma(n, R) U^+_\sigma(n, R) U^-_\sigma(n, R). \end{aligned}$$

θ-complete ring

By definition, we have

$$T'_\sigma(2n+1, R) \subseteq T_\sigma(2n+1, R).$$

Question

Under what conditions does the equality

$$T'_\sigma(2n+1, R) = T_\sigma(2n+1, R)$$

hold?

Define

$$\mathcal{B}_1(R) = \{u \in R \mid \exists t \in R \text{ such that } (t, u) \in \mathcal{A}(R)^\times\},$$
$$\mathcal{B}_k(R) = \{u \in R \mid \exists u_1, \ldots, u_k \in \mathcal{B}_1(R) \text{ such that } u = u_1 \cdots u_k\},$$
$$C_{even}(R) = \bigcup_{\ell \in \mathbb{N}} \mathcal{B}_{2\ell}(R).$$

θ-complete ring (continue)

Definition

Let R be a commutative ring with an involution θ.

1. We say that R is θ-complete if $\mathcal{C}_{even}(R) = R^{\times}$.
2. If $\exists k \in \mathbb{N}$ such that $\mathcal{C}_{even}(R) = \mathcal{B}_{2k}(R)$, then we say that R has finite $\mathcal{C}$-length. The minimal such k for which this equality holds is called the $\mathcal{C}$-length of R.

Example

- Every field is θ-complete with $\mathcal{C}$-length 1.
- Let R be a local ring. Assume that $2 \in R^{\times}$ and $R^{\times} \cap R_{\theta}^{-} \neq \emptyset$. Then R is θ-complete with $\mathcal{C}$-length ≤ 2.

Unitriangular Factorization of $SU(2n+1, R)$

Proposition

Let R be a θ-complete ring with $\mathcal{C}$-length k. Then

$$T_\sigma(2n{+}1, R) = T'_\sigma(2n{+}1, R) \subseteq (U^+_\sigma(2n{+}1, R)U^-_\sigma(2n{+}1, R))^{k+1}, \ (n \geq 1).$$

Theorem

Let R be a commutative ring with an involution θ. Assume that

1. *R is θ-complete with $\mathcal{C}$-length k and satisfies the (SSR_1) condition if $n = 1$,*
2. *R is θ-complete with $\mathcal{C}$-length k and satisfies both the (SR_1) and (SSR_1) condition if $n \geq 2$.*

Then

$$E'_\sigma(2n+1, R) = (U^+_\sigma(2n+1, R)U^-_\sigma(2n+1, R))^{k+1}U^+_\sigma(2n+1, R).$$

References

Shripad M. Garge and Deep H. Makadiya
Triangular and Unitriangular Factorization of Twisted Chevalley Groups, Accepted in Israel Journal of Mathematics.

Hyman Bass
K-theory and stable algebra, Institut des Hautes Études Scientifiques. Publications Mathématiques, No. 22 (1964), 5–60.

A. V. Smolensky, B. Sury and N. A. Vavilov
Unitriangular factorizations of Chevalley groups, Zapiski Nauchn. Semin. POMI, Vol. 388 (2011), 17–47.

References (continue)

A. V. Smolensky, B. Sury and N. A. Vavilov
Gauss decomposition for Chevalley groups, revisited, International Journal of Group Theory, Vol. 1 (2012), 3–16.

A. V. Smolensky
Unitriangular factorizations of twisted Chevalley groups, International Journal of Algebra and Computation, Vol. 23 (2013), 1497–1502.

O. I. Tavgen
Bounded generation of normal and twisted Chevalley groups over the rings of S-integers, Proceedings of the International Conference on Algebra, Part 1 (Novosibirsk, 1989), Vol. 131, Part 1 (1992), 409–421.

Thank You!