On Normal Subgroups of Twisted Chevalley Groups over Commutative Rings

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Introduction

One of the classical problems in the structure theory of groups is to determine all normal subgroups of a given group. We investigate this question for twisted Chevalley groups over commutative rings. In this setting, the standard approach focuses on describing subgroups that are normalized by the elementary subgroups (whether they are actually normalized by the whole twisted Chevalley group is a separate matter). Specifically, we address the following question:

What are all the subgroups of a twisted Chevalley group that are normalized by its elementary subgroups?

Chevalley Groups

It is known that semi-simple algebraic groups over $\mathbb{C}$ (or any algebraically closed field) are classified by their root systems (which determine their type) and the lattice between the root and fundamental weight lattices (which determines their isogeny class).

Let Φ be a root system, and let Λ_{ad} and Λ_{sc} denote the corresponding root lattice and fundamental weight lattice, respectively. Consider a lattice Λ_π satisfying $\Lambda_{\mathrm{ad}} \subseteq \Lambda_\pi \subseteq \Lambda_{sc}$. The unique (up to isomorphism) semisimple linear algebraic group over $\mathbb{C}$ associated with the pair (Φ, Λ_π) is denoted by $G_\pi(\Phi, \mathbb{C})$.

Moreover, this group is defined over $\mathbb{Z}$, allowing us to define the R-split group $G_\pi(\Phi, R)$ for any commutative ring R. This is known as the *Chevalley group* of type Φ over R.

For each root $\alpha \in \Phi$, there is an associated unipotent subgroup $U_\alpha = \{x_\alpha(t) \mid t \in R\} \subseteq G_\pi(\Phi, R)$, which is isomorphic to the additive group $(R, +)$. The group generated by all elements $x_\alpha(t)$ ($\alpha \in \Phi$ and $t \in R$) is called the *elementary Chevalley group*.

The following table provides some examples of classical Chevalley groups:

Type of Φ	Adjoint Groups	Simple Connected Groups
A_n $(n \geq 1)$	$PSL_{n+1}(R)$	$SL_{n+1}(R)$
B_n $(n \geq 2)$	$SO_{2n+1}(R) = PSO_{2n+1}(R)$	$Spin_{2n+1}(R)$
C_n $(n \geq 3)$	$PSp_{2n}(R)$	$Sp_{2n}(R)$
$D_n (n \geq 4)$	$PSO_{2n}(R)$	$Spin_{2n}(R)$

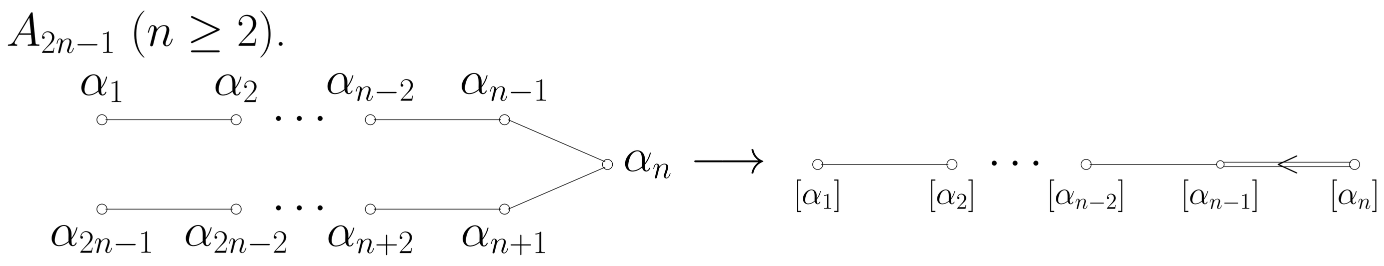
Twisted Root System

Assume that Φ is of type A_n $(n \geq 3)$, D_n $(n \geq 4)$, or E_6. Let ρ be a non-trivial angle-preserving permutation of simple roots. Note that the possible order of ρ is either 2 or 3, with the latter possible only when $\Phi \sim D_4$. Then this ρ induces an isometry of the ambient space $V = \mathrm{Span}_{\mathbb{R}}(\Phi)$, which we also denote by ρ. Define $\hat{\gamma} = \dfrac{\sum_{i=0}^{o(\rho)-1} \rho^i(\gamma)}{o(\rho)}$ for all $\gamma \in V$. Define an equivalent relation on Φ by setting $\alpha \equiv \beta$ if and only if $\hat{\alpha}$ is a positive multiple of $\hat{\beta}$. Let $[\alpha]$ denote the equivalence class of α and define Φ_ρ as the set of all equivalence classes. Then Φ_ρ also satisfies the axioms of the root system. Moreover, if $\Phi \sim X$ then we write $\Phi_\rho \sim {}^nX$ where n is the order of ρ.

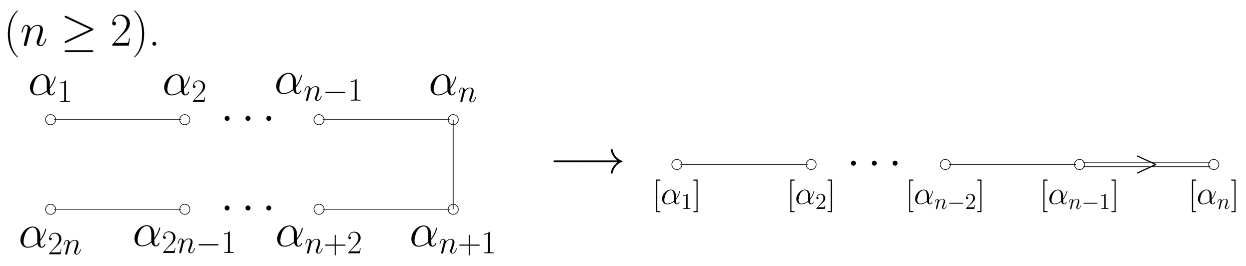
Type	${}^2A_{2n-1}$ $(n \geq 2)$	${}^2A_{2n}$ $(n \geq 2)$	${}^2D_n$ $(n \geq 4)$	${}^3D_4$	${}^2E_6$
Φ_ρ	C_n	B_n	B_{n-1}	G_2	F_4

For instance,

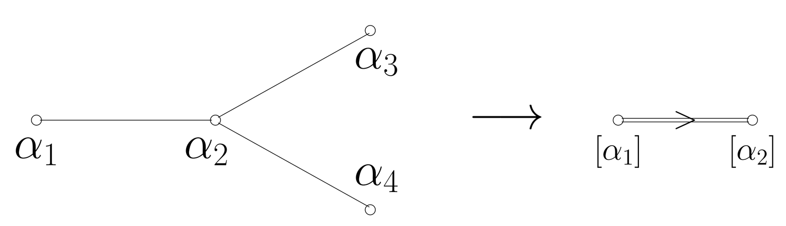
(a) $\Phi \sim A_{2n-1}$ $(n \geq 2)$.



(b) $\Phi \sim A_{2n}$ $(n \geq 2)$.



(c) $\Phi \sim D_4$.



Twisted Chevalley Groups

Let σ be a composition of a graph automorphism ρ and a ring automorphism θ, both of the same order (the possible orders are 2 or 3). Define $G_{\pi,\sigma}(\Phi, R)$ as the subgroup of $G_\pi(\Phi, R)$ of fixed points under the automorphism σ. This group is called the *twisted Chevalley group* of type Φ over R. Consider its elementary subgroup $E'_{\pi,\sigma}(\Phi, R)$ (i.e., the subgroup generated by $x_{[\alpha]}(t)$ where $[\alpha] \in \Phi_\rho$ and $t \in R_{[\alpha]}$), known as the *elementary twisted Chevalley group* of type Φ over R.

Congruence Subgroups

Let J be a θ-invariant ideal of a ring R. Consider the natural map

$$\phi : G_{\pi,\sigma}(\Phi, R) \longrightarrow G_{\pi,\sigma}(\Phi, R/J).$$

Define the subgroups

$$G_{\pi,\sigma}(\Phi, J) := \ker(\phi) \quad \text{and} \quad G_{\pi,\sigma}(\Phi, R, J) := \phi^{-1}(Z(G_{\pi,\sigma}(\Phi, R/J))).$$

The subgroup $G_{\pi,\sigma}(\Phi, J)$ of $G_{\pi,\sigma}(\Phi, R)$ is called a *principal congruence subgroup* of level J, while $G_{\pi,\sigma}(\Phi, R, J)$ is referred to as a *full congruence subgroup* of level J.

Let $E'_{\pi,\sigma}(\Phi, J)$ be the subgroup of $E'_{\pi,\sigma}(\Phi, R)$ generated by elements $x_{[\alpha]}(t)$ for all $[\alpha] \in \Phi_\rho$ and $t \in J_{[\alpha]}$. Additionally, define $E'_{\pi,\sigma}(\Phi, R, J)$, called the *relative elementary subgroup* of level J, as the normal subgroup of $E'_{\pi,\sigma}(\Phi, R)$ generated by $E'_{\pi,\sigma}(\Phi, J)$.

Main Results

Theorem 1 (Garge and Makadiya)**.** Let Φ_ρ be one of the following types: ${}^2A_n$ $(n \geq 3), {}^2D_n$ $(n \geq 4), {}^2E_6$, or ${}^3D_4$. Assume that $1/2 \in R$, and in addition, $1/3 \in R$ if $\Phi_\rho \sim {}^3D_4$. Then H is a subgroup of $G_{\pi,\sigma}(\Phi, R)$ normalized by $E'_{\pi,\sigma}(\Phi, R)$ if and only if there exists a unique θ-invariant ideal J of R such that

$$E'_{\pi,\sigma}(\Phi, R, J) \subseteq H \subseteq G_{\pi,\sigma}(\Phi, R, J).$$

Corollary 2. Let R and Φ_ρ be as in Theorem 1. Then a subgroup H of $E'_{\pi,\sigma}(\Phi, R)$ is normal in $E'_{\pi,\sigma}(\Phi, R)$ if and only if there exists a θ-invariant ideal J of R such that

$$E'_{\pi,\sigma}(\Phi, R, J) \subseteq H \subseteq G_{\pi,\sigma}(\Phi, R, J) \cap E'_{\pi,\sigma}(\Phi, R).$$

Definition. A subgroup H of G is said to be a *characteristic* if it is invariant under every automorphism of G.

As an application of the above theorem, we established the following result.

Theorem 3 (Garge and Makadiya)**.** *Let R and Φ_ρ be as in Theorem 1. Additionally, assume that R is a Noetherian ring. Let H be a subgroup of $G_{\pi,\sigma}(\Phi, R)$ containing $E'_{\pi,\sigma}(\Phi, R)$. Then $E'_{\pi,\sigma}(\Phi, R)$ is a characteristic subgroup of H. In particular, $E'_{\pi,\sigma}(\Phi, R)$ is a characteristic subgroup of $G_{\pi,\sigma}(\Phi, R)$.*

Remark. *We believe that this result holds even without assuming R to be Noetherian.*

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