Automorphisms of Chevalley Groups

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Conference on Groups and Representations

July 5, 2023

Outlines

1. Brief introduction to the Chevalley groups
2. Review some known results
3. Explore some special types of automorphisms
4. Twisted Chevalley groups and their automorphisms

Chevalley Groups

Let $\mathcal{L}$ be a semisimple Lie algebra over $\mathbb{C}$ with a root system Φ. Consider a Chevalley basis $\{X_\alpha, H_\beta \mid \alpha \in \Phi, \beta \in \Delta\}$.

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Let $\mathcal{U}$ be the universal enveloping algebra of $\mathcal{L}$ and let $\mathcal{U}_\mathbb{Z}$ be the Kostant's $\mathbb{Z}-$form generated by all $X_\alpha^m/m!$ $(m \in \mathbb{Z}_{\geq 0}; \alpha \in \Phi)$.

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Fact

V contains a lattice M invariant under $\mathcal{U}_\mathbb{Z}$, called an **admissible** lattice.

Let R be a commutative ring with unity. Define $V(R) = M \otimes_{\mathbb{Z}} R$.

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We wish to study automorphisms of $V(R)$ of the form $\exp(tX_\alpha)$ ($t \in R, \alpha \in \Phi$), where

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Write $x_\alpha(t)$ for $\exp(tX_\alpha)$ and $\mathfrak{X}_\alpha$ for the group $\{x_\alpha(t) \mid t \in R\}$.

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If π is a representation such that Λ_π contains all the fundamental dominant weights, then the corresponding group $G_{univ}(\mathcal{L}, R)$ is called a **universal Chevalley group**.

Examples

Type of Φ	G_{ad}	G_{univ}
A_n	PSL_{n+1}	SL_{n+1}
B_n	PSO_{n+1}	$Spin_{2n+1}$
C_n	PSp_{2n}	Sp_{2n}
D_n	PSO_{2n}	$Spin_{2n}$

Known Results

- **R. Steinberg[1] (1960)** classified all the automorphisms of the finite Chevalley groups over a field.

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- **E. Abe[3](1993)** further generalized it by considering Noetherian rings with unity.
- **E. I. Bunina[4](2012)** extended Abe's work by focusing on the automorphisms of adjoint Chevalley groups over commutative rings with unity.

[4] E. I. Bunina, Automorphisms of Chevalley groups of different types over commutative rings, J. Algebra, Volume 355, (2012), 154-170.

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(2) Field automorphisms. Let $\theta : k \longrightarrow k$ be an automorphism of the field k.

The mapping $(a_{ij}) \mapsto (\theta(a_{ij}))$ from G onto itself is an automorphism of the group G, that is denoted by the same letter θ and is called a *field automorphism* of the group G.

Note that for all $\alpha \in \Phi$ and $t \in k$, an element $x_\alpha(t) \mapsto x_\alpha(\theta(t))$.

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Let $g = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \in G$. Define $\psi = i_g \circ \phi$.

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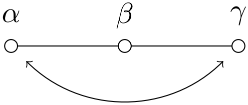
Then the image of $x_\alpha(t), x_\beta(t), x_\gamma(t)$ under this automorphism ψ is given by $x_\gamma(t), x_\beta(t), x_\alpha(t)$; respectively.

Observation:



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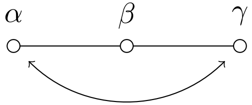
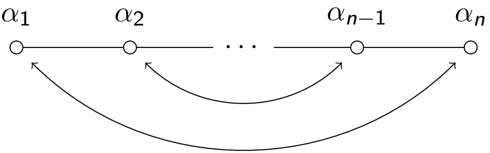


General case when $\Phi = A_n$:



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Then there exists a unique automorphism of G (we denote it by the same letter ρ) such that for every $\alpha \in \Phi$ and $t \in k$,

$$\rho(x_\alpha(t)) = \begin{cases} x_{\rho(\alpha)}(\epsilon(\alpha)t) & \text{if } \alpha \text{ is long or all roots are of one length,} \\ x_{\rho(\alpha)}(\epsilon(\alpha)t^p) & \text{if } \alpha \text{ is short.} \end{cases}$$

Here $\epsilon(\alpha) = \pm 1$ for all $\alpha \in \Phi$ and $\epsilon(\alpha) = 1$ for all $\alpha \in \Delta$.

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Here $\epsilon(\alpha) = \pm 1$ for all $\alpha \in \Phi$ and $\epsilon(\alpha) = 1$ for all $\alpha \in \Delta$.

Such an automorphism is called a **graph automorphism**.

(4) Diagonal automorphisms. Let $f_\alpha \in k^*$ for all simple α. Let f be the homomorphism from Λ_r into k^* given by $\alpha \mapsto f_\alpha$. Then there exists a unique automorphism ϕ of G such that $\phi x_\alpha(t) = x_\alpha(f_\alpha t)$ for all $\alpha \in \Phi$. An automorphism of this type is called *diagonal automorphism* of the group G.

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Theorem (Steinberg, Humphreys)

Let G be a Chevalley group such that Φ is indecomposable and k is perfect. Then any automorphism of group G can be expressed as the product of an inner, a graph, a diagonal and a field automorphism.

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(4′) Central automorphisms. Let $Z(G)$ be a center of $G = G_\pi(\Phi, R)$, $\tau : G \longrightarrow Z(G)$ be some homomorphism of groups. Then the mapping $x \mapsto \tau(x)x$ from G to itself is an automorphism of G, that is denoted by τ and called a *central automorphism* of the group G.

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Theorem (Bunina)

Let $G = G_{ad}(\Phi, R)$ be an adjoint Chevalley group over a commutative ring R with unity. Assume that Φ is indecomposable with rank > 1. Suppose that for $\Phi = A_2, B_l, C_l$ or F_4 we have $1/2 \in R$, for $\Phi = G_2$ we have $1/2, 1/3 \in R$. Then any automorphism of group G can be expressed as the product of an inner, a graph, a central and a ring automorphism.

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A subgroup $G'_\sigma = \langle U_\sigma, U_\sigma^- \rangle$ of G_σ is called a **twisted Chevalley group** over R.

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Theorem

Let G'_σ be a twisted Chevalley group over a perfect field k. Assume that σ is not the identity. Then every automorphism of G'_σ is a product of an inner, a diagonal, and a field automorphism.

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Remark

Observe that graph automorphisms are missing. Thus the twisted groups cannot themselves be twisted, at least not in the simple way we have been considering.

References

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Thank You!